Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Quiescent Optical Soliton Perturbation for Fokas-Lenells Equation with Nonlinear Chromatic Dispersion and Generalized Quadratic-Cubic Form of Self-Phase Modulation Structure

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Received: 6 January 2025; **Revised:** 14 March 2025; **Accepted:** 27 March 2025

Abstract: This paper retrieves perturbed quiescent optical solitons for the Fokas-Lenells equation that is considered with generalized quadratic-cubic form of self-phase modulation and nonlinear chromatic dispersion. The model is taken up with linear temporal evolution as well as with generalized temporal evolution. Two integration approaches are implemented to make this retrieval possible. The enhanced Kudryashov's approach and the projective Riccati equation scheme recovers a wide range of such solitons. The numerical scheme displays a few simulations to such solitons.

***Keywords*:** solitons, self-phase modulation, chromatic dispersion

MSC: 78A60

1. Introduction

One of the models that describe dispersive optical soliton propagation is the Fokas-Lenells equation [1]. This model replenishes the low count of chromatic dispersion (CD) with a couple of additional terms that meets the delicate dispersion-nonlinearity balance. This model therefore pays an important role in the propagation dynamics of optical solitons across intercontinental distances. The current paper discusses the soliton dynamics with Fokas-lenells equation in presence of a

DOI: https://doi.org/10.37256/cm.6220256359

few Hamiltonian type perturbation terms. The self-phase modulation (SPM) structure is taken to be the generalized version of quadratic-cubic form. The delicate balance between CD and SPM is the key ingredient to the stable propagation of solitons along trans-continental and trans-oceanic distances. However, once this balance is disrupted for some reason, this stable transmission of solitons is gravely impacted. The solitons could collapse and/or get stalled during its propagation thus leading to the aspect of quiescent optical solitons. The current paper will be addressing this issue with one of the unwanted features.

It may so happen that one or more physical or engineering flaws could lead to the CD being rendered to be nonlinear. This would lead to the formation of a quiescent soliton that is an unwanted feature during the information carrier propagation across the planet. Therefore, catastrophic consequences would ensue. The current paper addresses the formation of quiescent optical solitons for the Fokas-Lenells equation when CD is nonlinear. There are two forms of integration algorithms that are employed in the work to retrieve these quiescent optical solitons. They are enhanced Kudryashov's approach and the projective Riccati equation algorithm. The model is considered with linear temporal evolution as well as generalized temporal evolution to provide a tasty flavor to the project. The results are comprehensively derived and supplemented with numerical simulations that are exhibited in the rest of the paper.

The article is organized as follows: Section 2 introduces the two powerful analytical techniques utilized in this study. The enhanced Kudryashov method is presented as an effective tool for constructing exact soliton solutions by refining the classical Kudryashov approach to accommodate higher-order nonlinearities. The new projective Riccati equations algorithm is also introduced as a robust method for handling nonlinear differential equations, particularly those incorporating generalized nonlinearities and dispersive effects. The mathematical formulation and implementation of these techniques are detailed to demonstrate their effectiveness in retrieving diverse soliton solutions. Section 3 formulates the FLE under linear temporal evolution, incorporating quadratic-cubic nonlinearity, self-phase modulation, and nonlinear chromatic dispersion. The impact of these nonlinear terms on soliton formation and stability is examined through an analytical framework. The role of linear temporal evolution in modifying soliton profiles is discussed, particularly in relation to the balance between dispersion and nonlinearity required for stable propagation. Section 4 extends the analysis by incorporating generalized temporal evolution into the FLE. This section investigates how modifications to the temporal evolution parameter influence the behavior of optical solitons, particularly quiescent solitons. The combined effects of power-law nonlinearity, generalized quadratic-cubic SPM, and nonlinear CD are explored in detail. The analytical framework for solving the governing equation using the enhanced Kudryashov approach and the new projective Riccati equations algorithm is presented, leading to the derivation of novel soliton solutions. Section 5 presents a detailed analysis of the obtained soliton solutions through graphical illustrations. The physical interpretation of these results is provided, focusing on soliton amplitude variations, structural stability, and peak intensity modifications. The interplay between generalized temporal evolution, power-law nonlinearity, and quadratic-cubic SPM is discussed in the context of optical fiber transmission. Section 6 summarizes the key findings, emphasizing the significance of the newly derived soliton solutions in fiber optics. The study highlights the effectiveness of the enhanced Kudryashov approach and the new projective Riccati equations algorithm in obtaining diverse soliton structures under complex nonlinear and dispersive effects. The practical implications of these soliton solutions in optical communication systems are discussed, along with possible future extensions, such as the investigation of higher-dimensional models and additional perturbation effects.

2. Integration algorithms: an overview

We could possibly adopt a governance model that follows the structure of

$$F(q,\, q_x,\, q_t,\, q_{xt},\, q_{xx},\, \ldots) = 0. \tag{1}$$

The formula $q = q(x,\, t)$ denotes a waveform profile, with t and x representing temporal and spatial coordinates, respectively.

Utilizing the following wave transformation

$$q(x, t) = P(\zeta), \quad \zeta = k(x - \upsilon t), \tag{2}$$

yields a reduction of Eq. (1) to

$$G(P, -k\upsilon P', kP', k^2P'', \ldots) = 0. \tag{3}$$

In the given equation, k represents the wave width, ζ identifies the wave variable, and υ symbolizes the wave velocity.

2.1 *Enhanced Kudryashov's approach*

This subsection meticulously outlines the fundamentals of applying the enhanced Kudryashov's method [2, 3].

Step-1: We define the explicit solution for the simplified model Eq. (1) in the following manner:

$$P(\zeta) = \eta_0 + \sum_{i=1}^{N} \left\{ \eta_i R(\zeta)^i + \rho_i \left(\frac{R'(\zeta)}{R(\zeta)} \right)^i \right\}, \quad \eta_i^2 + \rho_i^2 \neq 0, \tag{4}$$

The auxiliary equation accompanies this,

$$R'(\zeta)^2 = R(\zeta)^2 (1 - \chi R(\zeta)^2), \tag{5}$$

Here, the constants η_0, χ, η_i, and ρ_i, for $i = 1, \ldots, N$, are determined, with N being ascertained by the balancing method in Eq. (3).

Step-2: From Eq. (5), we derive soliton waves as

$$R(\zeta) = \frac{4d\, e^{\zeta}}{4d^2 e^{2\zeta} + \chi}, \tag{6}$$

where d is a constant, Eq. (6) facilitates the generation of both bright solitons and singular solitons for $\chi = \pm 4d^2$, respectively.

Step-3: Substituting Eq. (4) into Eq. (3), and utilizing Eq. (5), enables the computation of essential constants for Eqs. (2) and (4). Incorporating the specified parametric constraints into Eq. (4), in conjunction with Eq. (6), yields a classification of solitons into bright, dark, or singular solitons.

2.2 *The new projective Riccati equations algorithm*

This section comprehensively outlines the essential phases of a cutting-edge approach known as the projective Riccati equations strategy and provides a detailed examination of its primary mechanisms [2, 3].

Step-1: It is postulated that a formal solution to Eq. (3) can be represented as follows:

$$P(\zeta) = \eta_0 + \sum_{i=1}^{N} \mho^{i-1}(\zeta)\left(\eta_i \mho(\zeta) + \rho_i \Omega(\zeta)\right), \tag{7}$$

The coefficients η_N and ρ_N mustn't be both zero at the same time. Moreover, the functions $\mho(\zeta)$ and $\Omega(\zeta)$ are specifically designed to satisfy the following ordinary differential equations (ODEs):

$$\mho'(\zeta) = -\mho(\zeta)\Omega(\zeta),$$

$$\Omega'(\zeta) = 1 - \Omega^2(\zeta) - \varpi\mho(\zeta), \tag{8}$$

accompanied by

$$\Omega(\zeta)^2 = 1 - 2\varpi\mho(\zeta) + R(\varpi)\mho(\zeta)^2, \tag{9}$$

The selection of the non-zero constant ϖ and the positive integer N is guided by the balancing regulation articulated in Eq. (3). This equation also integrates the real constants η_0, η_i, and ρ_i (for $i = 0, 1, ..., N$).

Step-2: The solutions to Eq. (8) are delineated as follows:

Case 1 $R(\varpi) = 0$.

$$\mho(\zeta) = \frac{1}{2\varpi}\operatorname{sech}^2\left[\frac{\zeta}{2}\right] \quad \text{and} \quad \Omega(\zeta) = \tanh\left[\frac{\zeta}{2}\right], \tag{10}$$

or

$$\mho(\zeta) = -\frac{1}{2\varpi}\operatorname{csch}^2\left[\frac{\zeta}{2}\right] \quad \text{and} \quad \Omega(\zeta) = \coth\left[\frac{\zeta}{2}\right]. \tag{11}$$

Case 2 $R(\varpi) = \frac{24}{25}\varpi^2$.

$$\mho(\zeta) = \frac{1}{\varpi}\frac{5\operatorname{sech}[\zeta]}{5\operatorname{sech}[\zeta] \pm 1} \quad \text{and} \quad \Omega(\zeta) = \frac{\pm\tanh[\zeta]}{5\operatorname{sech}[\zeta] \pm 1}. \tag{12}$$

Case 3 $R(\varpi) = \frac{5}{9}\varpi^2$.

$$\mho(\zeta) = \frac{1}{\varpi}\frac{3\operatorname{sech}[\zeta]}{3\operatorname{sech}[\zeta] \pm 2} \quad \text{and} \quad \Omega(\zeta) = \frac{2}{2\coth[\zeta] \pm 3\operatorname{csch}[\zeta]}. \tag{13}$$

Case 4 $R(\varpi) = \varpi^2 - 1$.

$$\mho(\zeta)=\frac{4\ \text{sech}[\zeta]}{3\ \tanh[\zeta]+4\varpi\ \text{sech}[\zeta]+5}\quad\text{and}\quad\Omega(\zeta)=\frac{5\ \tanh[\zeta]+3}{3\ \tanh[\zeta]+4\varpi\ \text{sech}[\zeta]+5},\tag{14}$$

or

$$\mho(\zeta)=\frac{\text{sech}[\zeta]}{\varpi\ \text{sech}[\zeta]+1}\quad\text{and}\quad\Omega(\zeta)=\frac{\tanh[\zeta]}{\varpi\ \text{sech}[\zeta]+1}.\tag{15}$$

Case 5 $R(\varpi)=\varpi^2+1$.

$$\mho(\zeta)=\frac{\text{csch}[\zeta]}{\varpi\ \text{csch}[\zeta]+1}\quad\text{and}\quad\Omega(\zeta)=\frac{\coth[\zeta]}{\varpi\ \text{csch}[\zeta]+1}.\tag{16}$$

Step-3: Integrating Eq. (7) alongside Eqs. (8) and (9), and then incorporating the resultant expression into Eq. (3), yields a polynomial equation characterized by the variables $\mho(\zeta)$ and $\Omega(\zeta)$, equating to zero. The coefficients of this polynomial, determined through this research, provide the critical parameters necessary for the application in Eqs. (2) and (7), as documented in existing studies.

3. Linear temporal evolution

3.1 *Generalized quadratic-cubic form*

The equation characterising the perturbed FLE with both nonlinear CD and generalized quadratic-cubic SPM in case of linear temporal evolution is provided by:

$$iq_t+a(|q|^n q)_{xx}+\left(b_1|q|^m+b_2|q|^{2m}\right)q+i\sigma|q|^2q_x=i\left[\alpha q_x+\lambda\left(|q|^2q\right)_x+\mu\left(|q|^2\right)_x q\right].\tag{17}$$

Within this framework, the expression $q=q(x,t)$ is a complex-valued function that characterizes the optical wave. In this context, x symbolizes the propagation distance within the optical medium, whereas t denotes the temporal variable. The term b_2 indicates the nonlinear coefficient, which reflects the self-interaction of the optical field due to the intensity-dependent refractive index of the medium, a phenomenon primarily attributed to the Kerr effect. This phenomenon occurs when the refractive index of the medium changes in relation to the intensity of the incident light. The nonlinear coefficient of chromatic dispersion (CD) is denoted by a, and the power-law nonlinearity in CD is introduced through the parameter n. The nonlinear dispersion is quantified by σ, while b_1 signifies the quadratic coefficient pertinent to self-phase modulation (SPM). The term iq_t encapsulates the temporal dynamics of the optical wave as it traverses the nonlinear medium. Additionally, α denotes the coefficient related to intermodal dispersion, and λ is associated with the self-steepening effect. The parameter μ is responsible for adjustments in self-frequency. Lastly, m is the full nonlinearity parameter.

The described profile corresponds to the quiescent optical soliton:

$$q(x,t)=P(\zeta)e^{i(\omega t+\theta)},\quad\zeta=kx,\tag{18}$$

Wherein $P(\zeta)$ denotes the amplitude, k represents the wave vector, ω signifies the frequency, and θ indicates the phase shift associated with the optical soliton. Upon applying the transformation delineated in Eq. (18), Eq. (17) is segregated into its constituent real and imaginary components.

$$ak^2(n+1)P^{n+1}P'' + ak^2n(n+1)P^nP'^2 + b_1P^{m+2} + b_2P^{2m+2} - \omega P^2 = 0. \tag{19}$$

$$-k\left(\alpha P' + (3\lambda + 2\mu - \sigma)P'P^2\right) = 0. \tag{20}$$

Based on the imaginary component, we can infer:

$$3\lambda + 2\mu - \sigma = 0, \quad \alpha = 0. \tag{21}$$

To fulfil the integrability criterion, we assume $m = n$. In this instance, Eq. (17) can be reformulated as:

$$iq_t + a(|q|^n q)_{xx} + \left(b_1 |q|^n + b_2 |q|^{2n}\right) q + i\sigma |q|^2 q_x = i\left[\lambda\left(|q|^2 q\right)_x + \mu\left(|q|^2\right)_x q\right], \tag{22}$$

and Eq. (19) can be rewritten as:

$$P^n\left(ak^2(n+1)\left(nP'^2 + PP''\right) + b_2P^{n+2} + b_1P^2\right) - \omega P^2 = 0. \tag{23}$$

Applying the transformation $P = Q^{2/n}$ to Eq. (23), leads to:

$$Q^2\left(-2ak^2\left(n^2+3n+2\right)Q'^2 - 2ak^2n(n+1)QQ'' + n^2\left(-b_2Q^4 - b_1Q^2 + \omega\right)\right) = 0. \tag{24}$$

Employing the balance principle in Eq. (24) results in $N = 1$. The subsequent subsections will implement the integration techniques on Eq. (24).

3.1.1 *Enhanced Kudryashov's approach*

According to the Enhanced Kudryashov method, the solution of Eq. (24) can be expressed in the following structure:

$$Q(\zeta) = \eta_0 + \eta_1 R(\zeta) + \rho_1\left(\frac{R'(\zeta)}{R(\zeta)}\right). \tag{25}$$

Substituting Eqs. (5) and (25) into Eq. (24) yields the subsequent array of algebraic equations:

$$\begin{aligned}&-24a\eta_1^2k^2\rho_1^2\chi^2 + 4a\eta_1^4k^2\chi - 36a\eta_1^2k^2n^2\rho_1^2\chi^2 + 6a\eta_1^4k^2n^2\chi + 6ak^2n^2\rho_1^4\chi^3 - 60a\eta_1^2k^2n\rho_1^2\chi^2\\ &+ 10a\eta_1^4k^2n\chi + 10ak^2n\rho_1^4\chi^3 + 4ak^2\rho_1^4\chi^3 - 15b_2\eta_1^2n^2\rho_1^4\chi^2 + 15b_2\eta_1^4n^2\rho_1^2\chi + b_2\eta_1^6\left(-n^2\right) + b_2n^2\rho_1^6\chi^3 = 0.\end{aligned} \tag{26}$$

$$-24a\eta_0\eta_1k^2\rho_1^2\chi^2+8a\eta_0\eta_1^3k^2\chi-48a\eta_0\eta_1k^2n^2\rho_1^2\chi^2+16a\eta_0\eta_1^3k^2n^2\chi-72a\eta_0\eta_1k^2n\rho_1^2\chi^2$$

$$+24a\eta_0\eta_1^3k^2n\chi-30b_2\eta_0\eta_1n^2\rho_1^4\chi^2+60b_2\eta_0\eta_1^3n^2\rho_1^2\chi-6b_2\eta_0\eta_1^5n^2=0. \tag{27}$$

$$-16a\eta_1k^2\rho_1^3\chi^2+16a\eta_1^3k^2\rho_1\chi-24a\eta_1k^2n^2\rho_1^3\chi^2+24a\eta_1^3k^2n^2\rho_1\chi-40a\eta_1k^2n\rho_1^3\chi^2$$

$$+40a\eta_1^3k^2n\rho_1\chi-6b_2\eta_1n^2\rho_1^5\chi^2+20b_2\eta_1^3n^2\rho_1^3\chi-6b_2\eta_1^5n^2\rho_1=0. \tag{28}$$

$$-4a\eta_0^2k^2\rho_1^2\chi^2+24a\eta_1^2k^2\rho_1^2\chi+4a\eta_0^2\eta_1^2k^2\chi-4a\eta_1^4k^2-14a\eta_0^2k^2n^2\rho_1^2\chi^2+42a\eta_1^2k^2n^2\rho_1^2\chi$$

$$+14a\eta_0^2\eta_1^2k^2n^2\chi-4a\eta_1^4k^2n^2-10ak^2n^2\rho_1^4\chi^2-18a\eta_0^2k^2n\rho_1^2\chi^2+66a\eta_1^2k^2n\rho_1^2\chi+18a\eta_0^2\eta_1^2k^2n\chi$$

$$-8a\eta_1^4k^2n-14ak^2n\rho_1^4\chi^2-4ak^2\rho_1^4\chi^2-15b_2\eta_0^2n^2\rho_1^4\chi^2+30b_2\eta_1^2n^2\rho_1^4\chi+90b_2\eta_0^2\eta_1^2n^2\rho_1^2\chi$$

$$+6b_1\eta_1^2n^2\rho_1^2\chi-15b_2\eta_1^4n^2\rho_1^2-15b_2\eta_0^2\eta_1^4n^2-b_1\eta_1^4n^2-3b_2n^2\rho_1^6\chi^2-b_1n^2\rho_1^4\chi^2=0. \tag{29}$$

$$-8a\eta_0k^2\rho_1^3\chi^2+24a\eta_0\eta_1^2k^2\rho_1\chi-16a\eta_0k^2n^2\rho_1^3\chi^2+48a\eta_0\eta_1^2k^2n^2\rho_1\chi-24a\eta_0k^2n\rho_1^3\chi^2$$

$$+72a\eta_0\eta_1^2k^2n\rho_1\chi-6b_2\eta_0n^2\rho_1^5\chi^2+60b_2\eta_0\eta_1^2n^2\rho_1^3\chi-30b_2\eta_0\eta_1^4n^2\rho_1=0. \tag{30}$$

$$16a\eta_0\eta_1k^2\rho_1^2\chi-8a\eta_0\eta_1^3k^2+50a\eta_0\eta_1k^2n^2\rho_1^2\chi+4a\eta_0^3\eta_1k^2n^2\chi-10a\eta_0\eta_1^3k^2n^2+66a\eta_0\eta_1k^2n\rho_1^2\chi$$

$$+4a\eta_0^3\eta_1k^2n\chi-18a\eta_0\eta_1^3k^2n+60b_2\eta_0\eta_1n^2\rho_1^4\chi+60b_2\eta_0^3\eta_1n^2\rho_1^2\chi+12b_1\eta_0\eta_1n^2\rho_1^2\chi$$

$$-60b_2\eta_0\eta_1^3n^2\rho_1^2-20b_2\eta_0^3\eta_1^3n^2-4b_1\eta_0\eta_1^3n^2=0. \tag{31}$$

$$8a\eta_1k^2\rho_1^3\chi+8a\eta_0^2\eta_1k^2\rho_1\chi-8a\eta_1^3k^2\rho_1+22a\eta_1k^2n^2\rho_1^3\chi+28a\eta_0^2\eta_1k^2n^2\rho_1\chi-10a\eta_1^3k^2n^2\rho_1$$

$$+30a\eta_1k^2n\rho_1^3\chi+36a\eta_0^2\eta_1k^2n\rho_1\chi-18a\eta_1^3k^2n\rho_1+12b_2\eta_1n^2\rho_1^5\chi+60b_2\eta_0^2\eta_1n^2\rho_1^3\chi$$

$$+4b_1\eta_1n^2\rho_1^3\chi-20b_2\eta_1^3n^2\rho_1^3-60b_2\eta_0^2\eta_1^3n^2\rho_1-4b_1\eta_1^3n^2\rho_1=0. \tag{32}$$

$$-4a\eta_1^2k^2\rho_1^2-4a\eta_0^2\eta_1^2k^2+12a\eta_0^2k^2n^2\rho_1^2\chi-8a\eta_1^2k^2n^2\rho_1^2-8a\eta_0^2\eta_1^2k^2n^2+4ak^2n^2\rho_1^4\chi+12a\eta_0^2k^2n\rho_1^2\chi$$

$$-12a\eta_1^2k^2n\rho_1^2-12a\eta_0^2\eta_1^2k^2n+4ak^2n\rho_1^4\chi+30b_2\eta_0^2n^2\rho_1^4\chi+15b_2\eta_0^4n^2\rho_1^2\chi+6b_1\eta_0^2n^2\rho_1^2\chi$$

$$-15b_2\eta_1^2n^2\rho_1^4-90b_2\eta_0^2\eta_1^2n^2\rho_1^2-6b_1\eta_1^2n^2\rho_1^2-15b_2\eta_0^4\eta_1^2n^2-6b_1\eta_0^2\eta_1^2n^2+3b_2n^2\rho_1^6\chi$$

$$+2b_1n^2\rho_1^4\chi+\eta_1^2n^2\omega-n^2\rho_1^2\chi\omega=0. \tag{33}$$

$$-8a\eta_0\eta_1^2k^2\rho_1+12a\eta_0k^2n^2\rho_1^3\chi+4a\eta_0^3k^2n^2\rho_1\chi-16a\eta_0\eta_1^2k^2n^2\rho_1+12a\eta_0k^2n\rho_1^3\chi+4a\eta_0^3k^2n\rho_1\chi$$

$$-24a\eta_0\eta_1^2k^2n\rho_1+12b_2\eta_0n^2\rho_1^5\chi+20b_2\eta_0^3n^2\rho_1^3\chi+4b_1\eta_0n^2\rho_1^3\chi-60b_2\eta_0\eta_1^2n^2\rho_1^3$$

$$-60b_2\eta_0^3\eta_1^2n^2\rho_1-12b_1\eta_0\eta_1^2n^2\rho_1=0. \tag{34}$$

$$-6a\eta_1\eta_0k^2n^2\rho_1^2-2a\eta_1\eta_0^3k^2n^2-6a\eta_1\eta_0k^2n\rho_1^2-2a\eta_1\eta_0^3k^2n-60b_2\eta_1\eta_0^3n^2\rho_1^2-30b_2\eta_1\eta_0n^2\rho_1^4$$

$$-12b_1\eta_1\eta_0n^2\rho_1^2-6b_2\eta_1\eta_0^5n^2-4b_1\eta_1\eta_0^3n^2+2\eta_1\eta_0n^2\omega=0. \tag{35}$$

$$-2a\eta_1k^2n^2\rho_1^3-6a\eta_0^2\eta_1k^2n^2\rho_1-2a\eta_1k^2n\rho_1^3-6a\eta_0^2\eta_1k^2n\rho_1-6b_2\eta_1n^2\rho_1^5-60b_2\eta_0^2\eta_1n^2\rho_1^3$$

$$-4b_1\eta_1n^2\rho_1^3-30b_2\eta_0^4\eta_1n^2\rho_1-12b_1\eta_0^2\eta_1n^2\rho_1+2\eta_1n^2\rho_1\omega=0. \tag{36}$$

$$-15b_2\eta_0^4n^2\rho_1^2-15b_2\eta_0^2n^2\rho_1^4-6b_1\eta_0^2n^2\rho_1^2+b_2\eta_0^6\left(-n^2\right)-b_1\eta_0^4n^2$$

$$-b_2n^2\rho_1^6-b_1n^2\rho_1^4+\eta_0^2n^2\omega+n^2\rho_1^2\omega=0. \tag{37}$$

$$-6b_2\eta_0^5n^2\rho_1-20b_2\eta_0^3n^2\rho_1^3-4b_1\eta_0^3n^2\rho_1-6b_2\eta_0n^2\rho_1^5-4b_1\eta_0n^2\rho_1^3+2\eta_0n^2\rho_1\omega=0. \tag{38}$$

The above equations provide the following subsequent results:

Result 1

$$k=\pm n\sqrt{\frac{b_1}{8\left(an^2+2an+a\right)}},\ \eta_0=0,\ \eta_1=0,\ \rho_1=\pm\sqrt{-\frac{b_1(3n+2)}{4b_2n+4b_2}},\ \omega=-\frac{b_1^2\left(3n^2+8n+4\right)}{16b_2(n+1)^2}. \tag{39}$$

Consequently, a solution of the governing Eq. (17) is achieved:

$$q(x,t)=\left\{\mp\sqrt{-\frac{b_1(3n+2)}{4b_2n+4b_2}}\frac{4d^2e^{\pm 2n\sqrt{\frac{b_1}{8\left(an^2+2an+a\right)}}}-\chi}{4d^2e^{\pm 2n\sqrt{\frac{b_1}{8\left(an^2+2an+a\right)}}}+\chi}\right\}^{\frac{2}{n}}e^{i\left(-\frac{b_1^2\left(3n^2+8n+4\right)}{16b_2(n+1)^2}t+\theta\right)}. \tag{40}$$

Choosing $\chi=\pm 4d^2$ to generate dark and singular solitons for the conditions $\frac{b_1(3n+2)}{4b_2n+4b_2}<0$ and $\frac{b_1}{(an^2+2ann+a)}$ >0:

$$q(x,t)=\left(\sqrt{-\frac{b_1(3n+2)}{4b_2n+4b_2}}\tanh\left(n\sqrt{\frac{b_1}{8\left(an^2+2an+a\right)}}x\right)\right)^{\frac{2}{n}}e^{i\left(-\frac{b_1^2\left(3n^2+8n+4\right)}{16b_2(n+1)^2}t+\theta\right)}, \tag{41}$$

$$q(x,t)=\left(\sqrt{-\frac{b_1(3n+2)}{4b_2n+4b_2}}\coth\left(n\sqrt{\frac{b_1}{8\left(an^2+2an+a\right)}}x\right)\right)^{\frac{2}{n}}e^{i\left(-\frac{b_1^2\left(3n^2+8n+4\right)}{16b_2(n+1)^2}t+\theta\right)}. \tag{42}$$

3.1.2 *New projective Riccati equation algorithm*

According to the novel projective Riccati equations methodology, the result can be expressed in the subsequent format.

$$Q(\zeta)=\eta_0+\eta_1\mho(\zeta)+\rho_1\Omega(\zeta). \tag{43}$$

By including Eq. (43) with Eqs. (8) and (9) into Eq. (24), the resultant system of algebraic equations is derived:

$$\begin{aligned}&-24a\eta_1^3k^2n^2\rho_1R(\varpi)-24a\eta_1k^2n^2\rho_1^3R(\varpi)^2-40a\eta_1^3k^2n\rho_1R(\varpi)-40a\eta_1k^2n\rho_1^3R(\varpi)^2\\&-16a\eta_1^3k^2\rho_1R(\varpi)-16a\eta_1k^2\rho_1^3R(\varpi)^2-6b_2\eta_1^5n^2\rho_1-20b_2\eta_1^3n^2\rho_1^3R(\varpi)-6b_2\eta_1n^2\rho_1^5R(\varpi)^2=0.\end{aligned} \tag{44}$$

$$\begin{aligned}&24a\eta_1^3k^2\rho_1\varpi+32a\eta_1^3k^2n^2\rho_1\varpi-16a\eta_0k^2n^2\rho_1^3R(\varpi)^2+64a\eta_1k^2n^2\rho_1^3\varpi R(\varpi)-48a\eta_0\eta_1^2k^2n^2\rho_1R(\varpi)\\&+56a\eta_1^3k^2n\rho_1\varpi-24a\eta_0k^2n\rho_1^3R(\varpi)^2+104a\eta_1k^2n\rho_1^3\varpi R(\varpi)-72a\eta_0\eta_1^2k^2n\rho_1R(\varpi)-8a\eta_0k^2\rho_1^3R(\varpi)^2\\&+40a\eta_1k^2\rho_1^3\varpi R(\varpi)-24a\eta_0\eta_1^2k^2\rho_1R(\varpi)+40b_2\eta_1^3n^2\rho_1^3\varpi-30b_2\eta_0\eta_1^4n^2\rho_1-6b_2\eta_0n^2\rho_1^5R(\varpi)^2\\&+24b_2\eta_1n^2\rho_1^5\varpi R(\varpi)-60b_2\eta_0\eta_1^2n^2\rho_1^3R(\varpi)=0.\end{aligned} \tag{45}$$

$$-24a\eta_1k^2\rho_1^3\varpi^2+32a\eta_0\eta_1^2k^2\rho_1\varpi-8a\eta_1^3k^2\rho_1-36a\eta_1k^2n^2\rho_1^3\varpi^2+58a\eta_0\eta_1^2k^2n^2\rho_1\varpi-10a\eta_1^3k^2n^2\rho_1$$

$$+38a\eta_0k^2n^2\rho_1^3\varpi R(\varpi)-22a\eta_1k^2n^2\rho_1^3R(\varpi)-28a\eta_0^2\eta_1k^2n^2\rho_1R(\varpi)-60a\eta_1k^2n\rho_1^3\varpi^2+90a\eta_0\eta_1^2k^2n\rho_1\varpi$$

$$-18a\eta_1^3k^2n\rho_1+54a\eta_0k^2n\rho_1^3\varpi R(\varpi)-30a\eta_1k^2n\rho_1^3R(\varpi)-36a\eta_0^2\eta_1k^2n\rho_1R(\varpi)+16a\eta_0k^2\rho_1^3\varpi R(\varpi)$$

$$-8a\eta_1k^2\rho_1^3R(\varpi)-8a\eta_0^2\eta_1k^2\rho_1R(\varpi)-24b_2\eta_1n^2\rho_1^5\varpi^2+120b_2\eta_0\eta_1^2n^2\rho_1^3\varpi-20b_2\eta_1^3n^2\rho_1^3$$

$$-60b_2\eta_0^2\eta_1^3n^2\rho_1-4b_1\eta_1^3n^2\rho_1+24b_2\eta_0n^2\rho_1^5\varpi R(\varpi)-12b_2\eta_1n^2\rho_1^5R(\varpi)$$

$$-60b_2\eta_0^2\eta_1n^2\rho_1^3R(\varpi)-4b_1\eta_1n^2\rho_1^3R(\varpi)=0. \tag{46}$$

$$-8a\eta_0k^2\rho_1^3\varpi^2+8a\eta_1k^2\rho_1^3\varpi+8a\eta_0^2\eta_1k^2\rho_1\varpi-8a\eta_0\eta_1^2k^2\rho_1-16a\eta_0k^2n^2\rho_1^3\varpi^2+20a\eta_1k^2n^2\rho_1^3\varpi$$

$$+28a\eta_0^2\eta_1k^2n^2\rho_1\varpi-16a\eta_0\eta_1^2k^2n^2\rho_1-12a\eta_0k^2n^2\rho_1^3R(\varpi)-4a\eta_0^3k^2n^2\rho_1R(\varpi)-24a\eta_0k^2n\rho_1^3\varpi^2$$

$$+28a\eta_1k^2n\rho_1^3\varpi+36a\eta_0^2\eta_1k^2n\rho_1\varpi-24a\eta_0\eta_1^2k^2n\rho_1-12a\eta_0k^2n\rho_1^3R(\varpi)-4a\eta_0^3k^2n\rho_1R(\varpi)$$

$$-24b_2\eta_0n^2\rho_1^5\varpi^2+24b_2\eta_1n^2\rho_1^5\varpi+120b_2\eta_0^2\eta_1n^2\rho_1^3\varpi+8b_1\eta_1n^2\rho_1^3\varpi-60b_2\eta_0\eta_1^2n^2\rho_1^3$$

$$-60b_2\eta_0^3\eta_1^2n^2\rho_1-12b_1\eta_0\eta_1^2n^2\rho_1-12b_2\eta_0n^2\rho_1^5R(\varpi)-20b_2\eta_0^3n^2\rho_1^3R(\varpi)-4b_1\eta_0n^2\rho_1^3R(\varpi)=0. \tag{47}$$

$$6a\eta_0k^2n^2\rho_1^3\varpi+2a\eta_0^3k^2n^2\rho_1\varpi-2a\eta_1k^2n^2\rho_1^3-6a\eta_0^2\eta_1k^2n^2\rho_1+6a\eta_0k^2n\rho_1^3\varpi+2a\eta_0^3k^2n\rho_1\varpi$$

$$-2a\eta_1k^2n\rho_1^3-6a\eta_0^2\eta_1k^2n\rho_1+24b_2\eta_0n^2\rho_1^5\varpi+40b_2\eta_0^3n^2\rho_1^3\varpi+8b_1\eta_0n^2\rho_1^3\varpi-6b_2\eta_1n^2\rho_1^5$$

$$-60b_2\eta_0^2\eta_1n^2\rho_1^3-4b_1\eta_1n^2\rho_1^3-30b_2\eta_0^4\eta_1n^2\rho_1-12b_1\eta_0^2\eta_1n^2\rho_1+2\eta_1n^2\rho_1\omega,\ \{1,0\}\to-6b_2\eta_0^5n^2\rho_1$$

$$-20b_2\eta_0^3n^2\rho_1^3-4b_1\eta_0^3n^2\rho_1-6b_2\eta_0n^2\rho_1^5-4b_1\eta_0n^2\rho_1^3+2\eta_0n^2\rho_1\omega=0. \tag{48}$$

$$-36a\eta_1^2k^2n^2\rho_1^2R(\varpi)^2-6a\eta_1^4k^2n^2R(\varpi)-6ak^2n^2\rho_1^4R(\varpi)^3-60a\eta_1^2k^2n\rho_1^2R(\varpi)^2-10a\eta_1^4k^2nR(\varpi)$$

$$-10ak^2n\rho_1^4R(\varpi)^3-24a\eta_1^2k^2\rho_1^2R(\varpi)^2-4a\eta_1^4k^2R(\varpi)-4ak^2\rho_1^4R(\varpi)^3+b_2\eta_1^6\left(-n^2\right)$$

$$-15b_2\eta_1^4n^2\rho_1^2R(\varpi)-15b_2\eta_1^2n^2\rho_1^4R(\varpi)^2-b_2n^2\rho_1^6R(\varpi)^3=0. \tag{49}$$

$$8a\eta_1^4k^2\varpi+10a\eta_1^4k^2n^2\varpi+108a\eta_1^2k^2n^2\rho_1^2\varpi R(\varpi)-48a\eta_0\eta_1k^2n^2\rho_1^2R(\varpi)^2-16a\eta_0\eta_1^3k^2n^2R(\varpi)$$

$$+26ak^2n^2\rho_1^4\varpi R(\varpi)^2+18a\eta_1^4k^2n\varpi+180a\eta_1^2k^2n\rho_1^2\varpi R(\varpi)-72a\eta_0\eta_1k^2n\rho_1^2R(\varpi)^2$$

$$-24a\eta_0\eta_1^3k^2nR(\varpi)+42ak^2n\rho_1^4\varpi R(\varpi)^2+72a\eta_1^2k^2\rho_1^2\varpi R(\varpi)-24a\eta_0\eta_1k^2\rho_1^2R(\varpi)^2$$

$$-8a\eta_0\eta_1^3k^2R(\varpi)+16ak^2\rho_1^4\varpi R(\varpi)^2+30b_2\eta_1^4n^2\rho_1^2\varpi-6b_2\eta_0\eta_1^5n^2+60b_2\eta_1^2n^2\rho_1^4\varpi R(\varpi)$$

$$-30b_2\eta_0\eta_1n^2\rho_1^4R(\varpi)^2-60b_2\eta_0\eta_1^3n^2\rho_1^2R(\varpi)+6b_2n^2\rho_1^6\varpi R(\varpi)^2=0. \tag{50}$$

$$-3n^2R(\varpi)^2b_2\rho_1^6-12n^2\varpi^2R(\varpi)b_2\rho_1^6-4ak^2R(\varpi)^2\rho_1^4-10ak^2n^2R(\varpi)^2\rho_1^4-14ak^2nR(\varpi)^2\rho_1^4$$

$$-15n^2R(\varpi)^2b_2\eta_0^2\rho_1^4-60n^2\varpi^2b_2\eta_1^2\rho_1^4-30n^2R(\varpi)b_2\eta_1^2\rho_1^4-20ak^2\varpi^2R(\varpi)\rho_1^4$$

$$-34ak^2n^2\varpi^2R(\varpi)\rho_1^4-54ak^2n\varpi^2R(\varpi)\rho_1^4-n^2R(\varpi)^2b_1\rho_1^4+120n^2\varpi R(\varpi)b_2\eta_0\eta_1\rho_1^4$$

$$-15n^2b_2\eta_1^4\rho_1^2+120n^2\varpi b_2\eta_0\eta_1^3\rho_1^2-4ak^2R(\varpi)^2\eta_0^2\rho_1^2-14ak^2n^2R(\varpi)^2\eta_0^2\rho_1^2$$

$$-18ak^2nR(\varpi)^2\eta_0^2\rho_1^2-52ak^2\varpi^2\eta_1^2\rho_1^2-74ak^2n^2\varpi^2\eta_1^2\rho_1^2-126ak^2n\varpi^2\eta_1^2\rho_1^2$$

$$-90n^2R(\varpi)b_2\eta_0^2\eta_1^2\rho_1^2-24ak^2R(\varpi)\eta_1^2\rho_1^2-42ak^2n^2R(\varpi)\eta_1^2\rho_1^2-66ak^2nR(\varpi)\eta_1^2\rho_1^2$$

$$-6n^2R(\varpi)b_1\eta_1^2\rho_1^2+64ak^2\varpi R(\varpi)\eta_0\eta_1\rho_1^2+134ak^2n^2\varpi R(\varpi)\eta_0\eta_1\rho_1^2$$

$$+198ak^2n\varpi R(\varpi)\eta_0\eta_1\rho_1^2-4ak^2\eta_1^4-4ak^2n^2\eta_1^4-15n^2b_2\eta_0^2\eta_1^4$$

$$-8ak^2n\eta_1^4-n^2b_1\eta_1^4+16ak^2\varpi\eta_0\eta_1^3+26ak^2n^2\varpi\eta_0\eta_1^3+42ak^2n\varpi\eta_0\eta_1^3$$

$$-4ak^2R(\varpi)\eta_0^2\eta_1^2-14ak^2n^2R(\varpi)\eta_0^2\eta_1^2-18ak^2nR(\varpi)\eta_0^2\eta_1^2=0. \tag{51}$$

$$-40a\eta_0\eta_1k^2\rho_1^2\varpi^2+32a\eta_1^2k^2\rho_1^2\varpi+8a\eta_0^2\eta_1^2k^2\varpi-8a\eta_0\eta_1^3k^2-80a\eta_0\eta_1k^2n^2\rho_1^2\varpi^2+52a\eta_1^2k^2n^2\rho_1^2\varpi$$

$$+22a\eta_0^2\eta_1^2k^2n^2\varpi-10a\eta_0\eta_1^3k^2n^2+12ak^2n^2\rho_1^4\varpi^3+34a\eta_0^2k^2n^2\rho_1^2\varpi R(\varpi)-50a\eta_0\eta_1k^2n^2\rho_1^2R(\varpi)$$

$$-4a\eta_0^3\eta_1k^2n^2R(\varpi)+24ak^2n^2\rho_1^4\varpi R(\varpi)-120a\eta_0\eta_1k^2n\rho_1^2\varpi^2+84a\eta_1^2k^2n\rho_1^2\varpi+30a\eta_0^2\eta_1^2k^2n\varpi$$

$$-18a\eta_0\eta_1^3k^2n+20ak^2n\rho_1^4\varpi^3+42a\eta_0^2k^2n\rho_1^2\varpi R(\varpi)-66a\eta_0\eta_1k^2n\rho_1^2R(\varpi)-4a\eta_0^3\eta_1k^2nR(\varpi)$$

$$+32ak^2n\rho_1^4\varpi R(\varpi)+8ak^2\rho_1^4\varpi^3+8a\eta_0^2k^2\rho_1^2\varpi R(\varpi)-16a\eta_0\eta_1k^2\rho_1^2R(\varpi)+8ak^2\rho_1^4\varpi R(\varpi)$$

$$-120b_2\eta_0\eta_1n^2\rho_1^4\varpi^2+60b_2\eta_1^2n^2\rho_1^4\varpi+180b_2\eta_0^2\eta_1^2n^2\rho_1^2\varpi+12b_1\eta_1^2n^2\rho_1^2\varpi-60b_2\eta_0\eta_1^3n^2\rho_1^2$$

$$-20b_2\eta_0^3\eta_1^3n^2-4b_1\eta_0\eta_1^3n^2+8b_2n^2\rho_1^6\varpi^3+60b_2\eta_0^2n^2\rho_1^4\varpi R(\varpi)-60b_2\eta_0\eta_1n^2\rho_1^4R(\varpi)$$

$$-60b_2\eta_0^3\eta_1n^2\rho_1^2R(\varpi)-12b_1\eta_0\eta_1n^2\rho_1^2R(\varpi)+12b_2n^2\rho_1^6\varpi R(\varpi)+4b_1n^2\rho_1^4\varpi R(\varpi)=0. \quad (52)$$

$$-4a\eta_0^2k^2\rho_1^2\varpi^2+16a\eta_0\eta_1k^2\rho_1^2\varpi-4a\eta_1^2k^2\rho_1^2-4a\eta_0^2\eta_1^2k^2-14a\eta_0^2k^2n^2\rho_1^2\varpi^2+50a\eta_0\eta_1k^2n^2\rho_1^2\varpi$$

$$-8a\eta_1^2k^2n^2\rho_1^2+6a\eta_0^3\eta_1k^2n^2\varpi-8a\eta_0^2\eta_1^2k^2n^2-10ak^2n^2\rho_1^4\varpi^2-12a\eta_0^2k^2n^2\rho_1^2R(\varpi)-4ak^2n^2\rho_1^4R(\varpi)$$

$$-18a\eta_0^2k^2n\rho_1^2\varpi^2+66a\eta_0\eta_1k^2n\rho_1^2\varpi-12a\eta_1^2k^2n\rho_1^2+6a\eta_0^3\eta_1k^2n\varpi-12a\eta_0^2\eta_1^2k^2n-14ak^2n\rho_1^4\varpi^2$$

$$-12a\eta_0^2k^2n\rho_1^2R(\varpi)-4ak^2n\rho_1^4R(\varpi)-4ak^2\rho_1^4\varpi^2-60b_2\eta_0^2n^2\rho_1^4\varpi^2+120b_2\eta_0\eta_1n^2\rho_1^4\varpi$$

$$+120b_2\eta_0^3\eta_1n^2\rho_1^2\varpi+24b_1\eta_0\eta_1n^2\rho_1^2\varpi-15b_2\eta_1^2n^2\rho_1^4-90b_2\eta_0^2\eta_1^2n^2\rho_1^2-6b_1\eta_1^2n^2\rho_1^2$$

$$-15b_2\eta_0^4\eta_1^2n^2-6b_1\eta_0^2\eta_1^2n^2-12b_2n^2\rho_1^6\varpi^2-4b_1n^2\rho_1^4\varpi^2-30b_2\eta_0^2n^2\rho_1^4R(\varpi)$$

$$-15b_2\eta_0^4n^2\rho_1^2R(\varpi)-6b_1\eta_0^2n^2\rho_1^2R(\varpi)-3b_2n^2\rho_1^6R(\varpi)$$

$$-2b_1n^2\rho_1^4R(\varpi)+\eta_1^2n^2\omega+n^2\rho_1^2\omega R(\varpi)=0. \quad (53)$$

$$6a\eta_0^2k^2n^2\rho_1^2\varpi-6a\eta_0\eta_1k^2n^2\rho_1^2-2a\eta_0^3\eta_1k^2n^2+2ak^2n^2\rho_1^4\varpi+6a\eta_0^2k^2n\rho_1^2\varpi-6a\eta_0\eta_1k^2n\rho_1^2$$

$$-2a\eta_0^3\eta_1k^2n+2ak^2n\rho_1^4\varpi+60b_2\eta_0^2n^2\rho_1^4\varpi+30b_2\eta_0^4n^2\rho_1^2\varpi+12b_1\eta_0^2n^2\rho_1^2\varpi$$

$$-30b_2\eta_0\eta_1n^2\rho_1^4-60b_2\eta_0^3\eta_1n^2\rho_1^2-12b_1\eta_0\eta_1n^2\rho_1^2-6b_2\eta_0^5\eta_1n^2-4b_1\eta_0^3\eta_1n^2$$

$$+6b_2n^2\rho_1^6\varpi+4b_1n^2\rho_1^4\varpi+2\eta_0\eta_1n^2\omega-2n^2\rho_1^2\varpi\omega=0. \tag{54}$$

$$-15b_2\eta_0^4n^2\rho_1^2-15b_2\eta_0^2n^2\rho_1^4-6b_1\eta_0^2n^2\rho_1^2+b_2\eta_0^6\left(-n^2\right)-b_1\eta_0^4n^2$$

$$-b_2n^2\rho_1^6-b_1n^2\rho_1^4+\eta_0^2n^2\omega+n^2\rho_1^2\omega=0. \tag{55}$$

The simultaneous computation of these equations produces the following results:
Case 1 $R(\varpi)=0$.

$$k=\pm n\sqrt{\frac{b_1}{2\left(an^2+2an+a\right)}},\ \eta_0=0,\ \eta_1=0,\ \rho_1=\pm\sqrt{\frac{-3b_1n-2b_1}{4\left(b_2n+b_2\right)}},\ \omega=-\frac{b_1^2\left(3n^2+8n+4\right)}{16b_2(n+1)^2}. \tag{56}$$

The closed-form solution to the governing model (17) is derived under the conditions that $\dfrac{-3b_1n-2b_1}{4\left(b_2n+b_2\right)}>0$ and $\dfrac{b_1}{(an^2+2ann+a)}>0$.

$$q(x,\,t)=\left(\sqrt{\frac{-3b_1n-2b_1}{4\left(b_2n+b_2\right)}}\tanh\left(n\sqrt{\frac{b_1}{8\left(an^2+2an+a\right)}}x\right)\right)^{\frac{2}{n}}e^{i\left(-\frac{b_1^2\left(3n^2+8n+4\right)}{16b_2(n+1)^2}t+\theta\right)}, \tag{57}$$

or

$$q(x,\,t)=\left(\sqrt{\frac{-3b_1n-2b_1}{4\left(b_2n+b_2\right)}}\coth\left(n\sqrt{\frac{b_1}{8\left(an^2+2an+a\right)}}x\right)\right)^{\frac{2}{n}}e^{i\left(-\frac{b_1^2\left(3n^2+8n+4\right)}{16b_2(n+1)^2}t+\theta\right)}. \tag{58}$$

Case 2 $R(\varpi)=\dfrac{24}{25}\varpi^2$.

$$k=\pm\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}},\ \eta_0=0,\ \eta_1=\pm\varpi\sqrt{-\frac{6b_1(3n+2)}{25b_2(n+1)}},$$

$$\rho_1=\pm\sqrt{-\frac{b_1(3n+2)}{4b_2(n+1)}},\ \omega=-\frac{b_1^2(n+2)(3n+2)}{16b_2(n+1)^2}. \tag{59}$$

As a result, the solution of the governing equation (17) for $\frac{b_1(3n+2)}{b_2(n+1)}<0$ and $\frac{b_1(3n+2)}{(n+1)\left(3an^2+5an+2a\right)}>0$ is attained:

$$q(x,t)=\left\{\pm\sqrt{-\frac{b_1(3n+2)}{b_2(n+1)}}\left(\frac{\sqrt{6}\operatorname{sech}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)}{5\operatorname{sech}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)\pm1}\right.\right.$$

$$\left.\left.\pm\frac{\tanh\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)}{2\left(1\pm5\operatorname{sech}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)\right)}\right)\right\}^{\frac{2}{n}}e^{i\left(-\frac{b_1^2(n+2)(3n+2)}{16b_2(n+1)^2}t+\theta\right)}. \tag{60}$$

Case 3 $R(\varpi)=\frac{5}{9}\varpi^2$.

$$k=\pm\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}},\ \eta_0=0,\ \eta_1=\pm\varpi\sqrt{-\frac{5b_1(3n+2)}{36b_2(n+1)}},$$

$$\rho_1=\pm\sqrt{-\frac{b_1(3n+2)}{4b_2(n+1)}},\ \omega=-\frac{b_1^2(n+2)(3n+2)}{16b_2(n+1)^2}. \tag{61}$$

The closed-form solution to the governing model (17) for $\frac{b_1(3n+2)}{b_2(n+1)}<0$ and $\frac{b_1(3n+2)}{(n+1)\left(3an^2+5an+2a\right)}>0$ is attained:

$$q(x,t)=\left\{\pm\sqrt{-\frac{b_1(3n+2)}{b_2(n+1)}}\left(\frac{1}{3\operatorname{csch}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)\pm2\coth\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)}\right.\right.$$

$$\left.\left.+\frac{\sqrt{5}\operatorname{sech}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)}{2\left(3\operatorname{sech}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)\pm2\right)}\right)\right\}^{\frac{2}{n}}e^{i\left(-\frac{b_1^2(n+2)(3n+2)}{16b_2(n+1)^2}t+\theta\right)}. \tag{62}$$

Case 4 $R(\varpi)=\varpi^2-1$.

$$k=\pm\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}},\ \eta_0=0,\ \eta_1=\pm\sqrt{-\frac{b_1(3n+2)\left(\varpi^2-1\right)}{4b_2(n+1)}},$$

$$\rho_1=\pm\sqrt{-\frac{b_1(3n+2)}{4b_2(n+1)}},\ \omega=-\frac{b_1^2(n+2)(3n+2)}{16b_2(n+1)^2}. \tag{63}$$

The closed-form solution to the governing model (17) for $\frac{b_1(3n+2)}{b_2(n+1)}<0$ and $\frac{b_1(3n+2)}{(n+1)\left(3an^2+5an+2a\right)}>0$ is attained:

$$q(x,t)=\left\{\pm\frac{1}{2}\sqrt{-\frac{b_1(3n+2)}{b_2(n+1)}}\right.$$

$$\left(\frac{4\sqrt{\varpi^2-1}\operatorname{sech}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)}{\left(4\varpi\operatorname{sech}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)\pm3\tanh\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)\right)+5}\right.$$

$$\left.\left.+\frac{3\pm5\tanh\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)}{\left(4\varpi\operatorname{sech}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)\pm3\tanh\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)\left(3an^2+5an+2a\right)}}x\right)\right)+5}\right)\right\}^{\frac{2}{n}}$$

$$\times e^{i\left(-\frac{b_1^2(n+2)(3n+2)}{16b_2(n+1)^2}t+\theta\right)}. \tag{64}$$

or

$$q(x,\,t)=\left\{\pm\frac{1}{2}\sqrt{-\frac{b_1(3n+2)}{b_2(n+1)}}\left(\frac{\sqrt{\varpi^2-1}\operatorname{sech}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)(3an^2+5an+2a)}}x\right)}{\varpi\operatorname{sech}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)(3an^2+5an+2a)}}x\right)+1}\right.\right.$$

$$\left.\left.\pm\frac{\tanh\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)(3an^2+5an+2a)}}x\right)}{\varpi\operatorname{sech}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)(3an^2+5an+2a)}}x\right)+1}\right)\right\}^{\frac{2}{n}}e^{i\left(-\frac{b_1^2(n+2)(3n+2)}{16b_2(n+1)^2}t+\theta\right)}. \tag{65}$$

Case 5 $R(\varpi)=\varpi^2+1$.

$$k=\pm\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)(3an^2+5an+2a)}},\;\eta_0=0,\;\eta_1=\pm\sqrt{-\frac{b_1(3n+2)(\varpi^2+1)}{4b_2(n+1)}},$$

$$\rho_1=\pm\sqrt{-\frac{b_1(3n+2)}{4b_2(n+1)}},\;\omega=-\frac{b_1^2(n+2)(3n+2)}{16b_2(n+1)^2}. \tag{66}$$

The closed-form solution to the governing model (17) for $\frac{b_1(3n+2)}{b_2(n+1)}<0$ and $\frac{b_1(3n+2)}{(n+1)(3an^2+5an+2a)}>0$ is attained:

$$q(x,\,t)=\left\{\pm\frac{1}{2}\sqrt{-\frac{b_1(3n+2)}{b_2(n+1)}}\left(\frac{\coth\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)(3an^2+5an+2a)}}x\right)}{1\pm\varpi\operatorname{csch}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)(3an^2+5an+2a)}}x\right)}\right.\right.$$

$$\left.\left.+\frac{\sqrt{\varpi^2+1}\operatorname{csch}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)(3an^2+5an+2a)}}x\right)}{1\pm\varpi\operatorname{csch}\left(\sqrt{\frac{b_1(3n+2)n^2}{2(n+1)(3an^2+5an+2a)}}x\right)}\right)\right\}^{\frac{2}{n}}e^{i\left(-\frac{b_1^2(n+2)(3n+2)}{16b_2(n+1)^2}t+\theta\right)}. \tag{67}$$

4. Generalized temporal evolution

4.1 *Generalized quadratic-cubic form*

The structure describing the perturbed FLE, incorporating both nonlinear CD and generalized quadratic-cubic SPM under generalized temporal evolution, is presented as follows:

$$i\left(q^{l}\right)_{t}+a(|q|^{n}q^{l})_{xx}+\left(b_1|q|^{m}+b_2|q|^{2m}\right)q^{l}+i\sigma|q|^{2}\left(q^{l}\right)_{x}=i\left[\alpha\left(q^{l}\right)_{x}+\lambda\left(|q|^{2}q^{l}\right)_{x}+\mu\left(|q|^{2}\right)_{x}q^{l}\right]. \tag{68}$$

where l provides the generalized temporal evolution. Utilizing the transformation (18) separates Eq. (68) into its respective real and imaginary components:

$$ak^{2}\left(l^{2}+l(2n-1)+(n-1)n\right)P^{n}P'^{2}+ak^{2}(l+n)P^{n+1}P''+b_1P^{m+2}+b_2P^{2m+2}-l\omega P^{2}=0. \tag{69}$$

$$-k\left((\lambda(l+2)-l\sigma+2\mu)v+\alpha lP'\right)=0. \tag{70}$$

From the imaginary part, we can deduce:

$$\lambda(l+2)-l\sigma+2\mu=0,\quad \alpha=0. \tag{71}$$

To fulfill the integrability criterion, we assume $m=n$. In this instance, Eq. (68) can be reformulated as:

$$i\left(q^{l}\right)_{t}+a(|q|^{n}q^{l})_{xx}+\left(b_1|q|^{n}+b_2|q|^{2n}\right)q^{l}+i\sigma|q|^{2}\left(q^{l}\right)_{x}=i\left[\lambda\left(|q|^{2}q^{l}\right)_{x}+\mu\left(|q|^{2}\right)_{x}q^{l}\right], \tag{72}$$

Eq. (69) can be reformulated as:

$$P^{n}\left(ak^{2}(l+n)\left((l+n-1)P'^{2}+PP''\right)+b_2P^{n+2}+b_1P^{2}\right)-l\omega P^{2}=0. \tag{73}$$

Applying the transformation $P=Q^{2/n}$ to Eq. (73), leads to:

$$Q^{2}\left(-2ak^{2}\left(2l^{2}+3ln+n^{2}\right)Q'^{2}-2ak^{2}n(l+n)QQ''+n^{2}\left(-b_2Q^{4}-b_1Q^{2}+l\omega\right)\right)=0. \tag{74}$$

Employing the balance principle in relation to Eq. (74) results in $N=1$. The subsequent subsections will employ the integration techniques on Eq. (74).

4.1.1 *Enhanced Kudryashov's approach*

Utilizing the enhanced Kudryashov's technique, the solution to Eq. (74) can be formulated in the subsequent form:

$$Q(\zeta)=\eta_0+\eta_1R(\zeta)+\rho_1\left(\frac{R'(\zeta)}{R(\zeta)}\right). \tag{75}$$

By substituting Eq. (75) and Eq. (5) into Eq. (74), the resultant system of algebraic equations is derived:

$$-24a\eta_1^2k^2l^2\rho_1^2\chi^2+4a\eta_1^4k^2l^2\chi+4ak^2l^2\rho_1^4\chi^3-60a\eta_1^2k^2ln\rho_1^2\chi^2+10a\eta_1^4k^2ln\chi$$

$$+10ak^2ln\rho_1^4\chi^3-36a\eta_1^2k^2n^2\rho_1^2\chi^2+6a\eta_1^4k^2n^2\chi+6ak^2n^2\rho_1^4\chi^3-15b_2\eta_1^2n^2\rho_1^4\chi^2$$

$$+15b_2\eta_1^4n^2\rho_1^2\chi+b_2\eta_1^6\left(-n^2\right)+b_2n^2\rho_1^6\chi^3=0. \tag{76}$$

$$-24a\eta_0\eta_1k^2l^2\rho_1^2\chi^2+8a\eta_0\eta_1^3k^2l^2\chi-72a\eta_0\eta_1k^2ln\rho_1^2\chi^2+24a\eta_0\eta_1^3k^2ln\chi-48a\eta_0\eta_1k^2n^2\rho_1^2\chi^2$$

$$+16a\eta_0\eta_1^3k^2n^2\chi-30b_2\eta_0\eta_1n^2\rho_1^4\chi^2+60b_2\eta_0\eta_1^3n^2\rho_1^2\chi-6b_2\eta_0\eta_1^5n^2=0. \tag{77}$$

$$-16a\eta_1k^2l^2\rho_1^3\chi^2+16a\eta_1^3k^2l^2\rho_1\chi-40a\eta_1k^2ln\rho_1^3\chi^2+40a\eta_1^3k^2ln\rho_1\chi-24a\eta_1k^2n^2\rho_1^3\chi^2$$

$$+24a\eta_1^3k^2n^2\rho_1\chi-6b_2\eta_1n^2\rho_1^5\chi^2+20b_2\eta_1^3n^2\rho_1^3\chi-6b_2\eta_1^5n^2\rho_1=0. \tag{78}$$

$$-4a\eta_0^2k^2l^2\rho_1^2\chi^2+24a\eta_1^2k^2l^2\rho_1^2\chi+4a\eta_0^2\eta_1^2k^2l^2\chi-4a\eta_1^4k^2l^2-4ak^2l^2\rho_1^4\chi^2$$

$$-18a\eta_0^2k^2ln\rho_1^2\chi^2+66a\eta_1^2k^2ln\rho_1^2\chi+18a\eta_0^2\eta_1^2k^2ln\chi-8a\eta_1^4k^2ln-14ak^2ln\rho_1^4\chi^2$$

$$-14a\eta_0^2k^2n^2\rho_1^2\chi^2+42a\eta_1^2k^2n^2\rho_1^2\chi+14a\eta_0^2\eta_1^2k^2n^2\chi-4a\eta_1^4k^2n^2-10ak^2n^2\rho_1^4\chi^2$$

$$-15b_2\eta_0^2n^2\rho_1^4\chi^2\chi+30b_2\eta_1^2n^2\rho_1^4\chi+90b_2\eta_0^2\eta_1^2n^2\rho_1^2+6b_1\eta_1^2n^2\rho_1^2\chi$$

$$-15b_2\eta_1^4n^2\rho_1^2-15b_2\eta_0^2\eta_1^4n^2-b_1\eta_1^4n^2-3b_2n^2\rho_1^6\chi^2-b_1n^2\rho_1^4\chi^2=0. \tag{79}$$

$$-8a\eta_0k^2l^2\rho_1^3\chi^2+24a\eta_0\eta_1^2k^2l^2\rho_1\chi-24a\eta_0k^2ln\rho_1^3\chi^2+72a\eta_0\eta_1^2k^2ln\rho_1\chi-16a\eta_0k^2n^2\rho_1^3\chi^2$$

$$+48a\eta_0\eta_1^2k^2n^2\rho_1\chi-6b_2\eta_0n^2\rho_1^5\chi^2+60b_2\eta_0\eta_1^2n^2\rho_1^3\chi-30b_2\eta_0\eta_1^4n^2\rho_1=0. \tag{80}$$

$$16a\eta_0\eta_1k^2l^2\rho_1^2\chi-8a\eta_0\eta_1^3k^2l^2+66a\eta_0\eta_1k^2ln\rho_1^2\chi+4a\eta_0^3\eta_1k^2ln\chi-18a\eta_0\eta_1^3k^2ln$$

$$+50a\eta_0\eta_1k^2n^2\rho_1^2\chi+4a\eta_0^3\eta_1k^2n^2\chi-10a\eta_0\eta_1^3k^2n^2+60b_2\eta_0\eta_1n^2\rho_1^4\chi+60b_2\eta_0^3\eta_1n^2\rho_1^2\chi$$

$$+12b_1\eta_0\eta_1n^2\rho_1^2\chi-60b_2\eta_0\eta_1^3n^2\rho_1^2-20b_2\eta_0^3\eta_1^3n^2-4b_1\eta_0\eta_1^3n^2=0. \tag{81}$$

$$8a\eta_1k^2l^2\rho_1^3\chi+8a\eta_0^2\eta_1k^2l^2\rho_1\chi-8a\eta_1^3k^2l^2\rho_1+30a\eta_1k^2ln\rho_1^3\chi+36a\eta_0^2\eta_1k^2ln\rho_1\chi-18a\eta_1^3k^2ln\rho_1$$

$$+22a\eta_1k^2n^2\rho_1^3\chi+28a\eta_0^2\eta_1k^2n^2\rho_1\chi-10a\eta_1^3k^2n^2\rho_1+12b_2\eta_1n^2\rho_1^5\chi+60b_2\eta_0^2\eta_1n^2\rho_1^3\chi$$

$$+4b_1\eta_1n^2\rho_1^3\chi-20b_2\eta_1^3n^2\rho_1^3-60b_2\eta_0^2\eta_1^3n^2\rho_1-4b_1\eta_1^3n^2\rho_1=0. \tag{82}$$

$$-4a\eta_1^2k^2l^2\rho_1^2-4a\eta_0^2\eta_1^2k^2l^2+12a\eta_0^2k^2ln\rho_1^2\chi-12a\eta_1^2k^2ln\rho_1^2-12a\eta_0^2\eta_1^2k^2ln+4ak^2ln\rho_1^4\chi$$

$$+12a\eta_0^2k^2n^2\rho_1^2\chi-8a\eta_1^2k^2n^2\rho_1^2-8a\eta_0^2\eta_1^2k^2n^2+4ak^2n^2\rho_1^4\chi+30b_2\eta_0^2n^2\rho_1^4\chi+15b_2\eta_0^4n^2\rho_1^2\chi$$

$$+6b_1\eta_0^2n^2\rho_1^2\chi-15b_2\eta_1^2n^2\rho_1^4-90b_2\eta_0^2\eta_1^2n^2\rho_1^2-6b_1\eta_1^2n^2\rho_1^2-15b_2\eta_0^4\eta_1^2n^2-6b_1\eta_0^2\eta_1^2n^2$$

$$+3b_2n^2\rho_1^6\chi+2b_1n^2\rho_1^4\chi+\eta_1^2ln^2\omega-ln^2\rho_1^2\chi\omega=0. \tag{83}$$

$$-8a\eta_0\eta_1^2k^2l^2\rho_1+12a\eta_0k^2ln\rho_1^3\chi+4a\eta_0^3k^2ln\rho_1\chi-24a\eta_0\eta_1^2k^2ln\rho_1+12a\eta_0k^2n^2\rho_1^3\chi$$

$$+4a\eta_0^3k^2n^2\rho_1\chi-16a\eta_0\eta_1^2k^2n^2\rho_1+12b_2\eta_0n^2\rho_1^5\chi+20b_2\eta_0^3n^2\rho_1^3\chi+4b_1\eta_0n^2\rho_1^3\chi$$

$$-60b_2\eta_0\eta_1^2n^2\rho_1^3-60b_2\eta_0^3\eta_1^2n^2\rho_1-12b_1\eta_0\eta_1^2n^2\rho_1=0. \tag{84}$$

$$-6a\eta_1\eta_0k^2ln\rho_1^2-2a\eta_1\eta_0^3k^2ln-6a\eta_1\eta_0k^2n^2\rho_1^2-2a\eta_1\eta_0^3k^2n^2-60b_2\eta_1\eta_0^3n^2\rho_1^2$$

$$-30b_2\eta_1\eta_0n^2\rho_1^4-12b_1\eta_1\eta_0n^2\rho_1^2-6b_2\eta_1\eta_0^5n^2-4b_1\eta_1\eta_0^3n^2+2\eta_1\eta_0ln^2\omega=0. \tag{85}$$

$$-2a\eta_1k^2ln\rho_1^3-6a\eta_0^2\eta_1k^2ln\rho_1-2a\eta_1k^2n^2\rho_1^3-6a\eta_0^2\eta_1k^2n^2\rho_1-6b_2\eta_1n^2\rho_1^5-60b_2\eta_0^2\eta_1n^2\rho_1^3$$

$$-4b_1\eta_1n^2\rho_1^3-30b_2\eta_0^4\eta_1n^2\rho_1-12b_1\eta_0^2\eta_1n^2\rho_1+2\eta_1ln^2\rho_1\omega=0. \tag{86}$$

$$-15b_2\eta_0^4n^2\rho_1^2-15b_2\eta_0^2n^2\rho_1^4-6b_1\eta_0^2n^2\rho_1^2+b_2\eta_0^6\left(-n^2\right)-b_1\eta_0^4n^2-b_2n^2\rho_1^6$$

$$-b_1n^2\rho_1^4+\eta_0^2ln^2\omega+ln^2\rho_1^2\omega=0. \tag{87}$$

$$-6b_2\eta_0^5n^2\rho_1-20b_2\eta_0^3n^2\rho_1^3-4b_1\eta_0^3n^2\rho_1-6b_2\eta_0n^2\rho_1^5-4b_1\eta_0n^2\rho_1^3+2\eta_0ln^2\rho_1\omega=0. \tag{88}$$

The simultaneous computation of these equations produces the following results:

Result 1

$$k=\pm n\sqrt{\frac{b_1}{8\left(al^2+2aln+an^2\right)}},\ \eta_0=0,\ \eta_1=0,\ \rho_1=\pm\sqrt{-\frac{b_1(2l+3n)}{4b_2l+4b_2n}},\ \omega=-\frac{b_1^2\left(4l^2+8ln+3n^2\right)}{16b_2l(l+n)^2}. \tag{89}$$

Consequently, the closed-form solution to the governing model Eq. (68) is achieved:

$$q(x,t)=\left\{\mp\sqrt{-\frac{b_1(2l+3n)}{4b_2l+4b_2n}}\frac{4d^2e^{\pm 2n\sqrt{\frac{b_1}{8\left(al^2+2aln+an^2\right)}}}-\chi}{4d^2e^{\pm 2n\sqrt{\frac{b_1}{8\left(al^2+2aln+an^2\right)}}}+\chi}\right\}^{\frac{2}{n}}e^{i\left(-\frac{b_1^2\left(4l^2+8ln+3n^2\right)}{16b_2l(l+n)^2}t+\theta\right)}, \tag{90}$$

Selecting $\chi=\pm 4d^2$ to recover dark and singular solitons for $\frac{b_1(2l+3n)}{4b_2l+4b_2n}<0$ and $\frac{b_1}{\left(al^2+2aln+an^2\right)}>0$:

$$q(x,t)=\left(\sqrt{-\frac{b_1(2l+3n)}{4b_2l+4b_2n}}\tanh\left(n\sqrt{\frac{b_1}{8\left(al^2+2aln+an^2\right)}}x\right)\right)^{\frac{2}{n}}e^{i\left(-\frac{b_1^2\left(4l^2+8ln+3n^2\right)}{16b_2l(l+n)^2}t+\theta\right)}, \tag{91}$$

$$q(x,t)\left(\sqrt{-\frac{b_1(2l+3n)}{4b_2l+4b_2n}}\coth\left(n\sqrt{\frac{b_1}{8\left(al^2+2aln+an^2\right)}}x\right)\right)^{\frac{2}{n}}e^{i\left(-\frac{b_1^2\left(4l^2+8ln+3n^2\right)}{16b_2l(l+n)^2}t+\theta\right)}. \tag{92}$$

4.1.2 *New projective Riccati equation algorithm*

According to the novel projective Riccati equations strategy, the result is articulated in the following format.

$$Q(\zeta)=\eta_0+\eta_1\mho(\zeta)+\rho_1\Omega(\zeta). \tag{93}$$

By including Eq. (93) with Eqs. (8) and (9) into Eq. (74), the resultant system of algebraic equations is derived:

$$\begin{aligned}&-16a\eta_1^3k^2l^2\rho_1R(\varpi)-16a\eta_1k^2l^2\rho_1^3R(\varpi)^2-40a\eta_1^3k^2ln\rho_1R(\varpi)-40a\eta_1k^2ln\rho_1^3R(\varpi)^2\\&-24a\eta_1^3k^2n^2\rho_1R(\varpi)-24a\eta_1k^2n^2\rho_1^3R(\varpi)^2-6b_2\eta_1^5n^2\rho_1\\&-20b_2\eta_1^3n^2\rho_1^3R(\varpi)-6b_2\eta_1n^2\rho_1^5R(\varpi)^2=0.\end{aligned} \tag{94}$$

$$24a\eta_1^3k^2l^2\rho_1\varpi - 8a\eta_0k^2l^2\rho_1^3R(\varpi)^2 + 40a\eta_1k^2l^2\rho_1^3\varpi R(\varpi) - 24a\eta_0\eta_1^2k^2l^2\rho_1R(\varpi) + 56a\eta_1^3k^2ln\rho_1\varpi$$

$$-24a\eta_0k^2ln\rho_1^3R(\varpi)^2 + 104a\eta_1k^2ln\rho_1^3\varpi R(\varpi) - 72a\eta_0\eta_1^2k^2ln\rho_1R(\varpi) + 32a\eta_1^3k^2n^2\rho_1\varpi$$

$$-16a\eta_0k^2n^2\rho_1^3R(\varpi)^2 + 64a\eta_1k^2n^2\rho_1^3\varpi R(\varpi) - 48a\eta_0\eta_1^2k^2n^2\rho_1R(\varpi) + 40b_2\eta_1^3n^2\rho_1^3\varpi$$

$$-30b_2\eta_0\eta_1^4n^2\rho_1 - 6b_2\eta_0n^2\rho_1^5R(\varpi)^2 + 24b_2\eta_1n^2\rho_1^5\varpi R(\varpi) - 60b_2\eta_0\eta_1^2n^2\rho_1^3R(\varpi) = 0. \tag{95}$$

$$-24a\eta_1k^2l^2\rho_1^3\varpi^2 + 32a\eta_0\eta_1^2k^2l^2\rho_1\varpi - 8a\eta_1^3k^2l^2\rho_1 + 16a\eta_0k^2l^2\rho_1^3\varpi R(\varpi) - 8a\eta_1k^2l^2\rho_1^3R(\varpi)$$

$$-8a\eta_0^2\eta_1k^2l^2\rho_1R(\varpi) - 60a\eta_1k^2ln\rho_1^3\varpi^2 + 90a\eta_0\eta_1^2k^2ln\rho_1\varpi - 18a\eta_1^3k^2ln\rho_1 + 54a\eta_0k^2ln\rho_1^3\varpi R(\varpi)$$

$$-30a\eta_1k^2ln\rho_1^3R(\varpi) - 36a\eta_0^2\eta_1k^2ln\rho_1R(\varpi) - 36a\eta_1k^2n^2\rho_1^3\varpi^2 + 58a\eta_0\eta_1^2k^2n^2\rho_1\varpi - 10a\eta_1^3k^2n^2\rho_1$$

$$+38a\eta_0k^2n^2\rho_1^3\varpi R(\varpi) - 22a\eta_1k^2n^2\rho_1^3R(\varpi) - 28a\eta_0^2\eta_1k^2n^2\rho_1R(\varpi) - 24b_2\eta_1n^2\rho_1^5\varpi^2$$

$$+120b_2\eta_0\eta_1^2n^2\rho_1^3\varpi - 20b_2\eta_1^3n^2\rho_1^3 - 60b_2\eta_0^2\eta_1^3n^2\rho_1 - 4b_1\eta_1^3n^2\rho_1 + 24b_2\eta_0n^2\rho_1^5\varpi R(\varpi)$$

$$-12b_2\eta_1n^2\rho_1^5R(\varpi) - 60b_2\eta_0^2\eta_1n^2\rho_1^3R(\varpi) - 4b_1\eta_1n^2\rho_1^3R(\varpi) = 0. \tag{96}$$

$$-8a\eta_0k^2l^2\rho_1^3\varpi^2 + 8a\eta_1k^2l^2\rho_1^3\varpi + 8a\eta_0^2\eta_1k^2l^2\rho_1\varpi - 8a\eta_0\eta_1^2k^2l^2\rho_1 - 24a\eta_0k^2ln\rho_1^3\varpi^2 + 28a\eta_1k^2ln\rho_1^3\varpi$$

$$+36a\eta_0^2\eta_1k^2ln\rho_1\varpi - 24a\eta_0\eta_1^2k^2ln\rho_1 - 12a\eta_0k^2ln\rho_1^3R(\varpi) - 4a\eta_0^3k^2ln\rho_1R(\varpi) - 16a\eta_0k^2n^2\rho_1^3\varpi^2$$

$$+20a\eta_1k^2n^2\rho_1^3\varpi + 28a\eta_0^2\eta_1k^2n^2\rho_1\varpi - 16a\eta_0\eta_1^2k^2n^2\rho_1 - 12a\eta_0k^2n^2\rho_1^3R(\varpi) - 4a\eta_0^3k^2n^2\rho_1R(\varpi)$$

$$-24b_2\eta_0n^2\rho_1^5\varpi^2 + 24b_2\eta_1n^2\rho_1^5\varpi + 120b_2\eta_0^2\eta_1n^2\rho_1^3\varpi + 8b_1\eta_1n^2\rho_1^3\varpi - 60b_2\eta_0\eta_1^2n^2\rho_1^3$$

$$-60b_2\eta_0^3\eta_1^2n^2\rho_1 - 12b_1\eta_0\eta_1^2n^2\rho_1 - 12b_2\eta_0n^2\rho_1^5R(\varpi) - 20b_2\eta_0^3n^2\rho_1^3R(\varpi) - 4b_1\eta_0n^2\rho_1^3R(\varpi) = 0. \tag{97}$$

$$6a\eta_0k^2ln\rho_1^3\varpi + 2a\eta_0^3k^2ln\rho_1\varpi - 2a\eta_1k^2ln\rho_1^3 - 6a\eta_0^2\eta_1k^2ln\rho_1 + 6a\eta_0k^2n^2\rho_1^3\varpi + 2a\eta_0^3k^2n^2\rho_1\varpi$$

$$-2a\eta_1k^2n^2\rho_1^3 - 6a\eta_0^2\eta_1k^2n^2\rho_1 + 24b_2\eta_0n^2\rho_1^5\varpi + 40b_2\eta_0^3n^2\rho_1^3\varpi + 8b_1\eta_0n^2\rho_1^3\varpi - 6b_2\eta_1n^2\rho_1^5$$

$$-60b_2\eta_0^2\eta_1n^2\rho_1^3 - 4b_1\eta_1n^2\rho_1^3 - 30b_2\eta_0^4\eta_1n^2\rho_1 - 12b_1\eta_0^2\eta_1n^2\rho_1 + 2\eta_1ln^2\rho_1\omega = 0. \tag{98}$$

$$-6b_2\eta_0^5n^2\rho_1 - 20b_2\eta_0^3n^2\rho_1^3 - 4b_1\eta_0^3n^2\rho_1 - 6b_2\eta_0n^2\rho_1^5 - 4b_1\eta_0n^2\rho_1^3 + 2\eta_0ln^2\rho_1\omega = 0. \tag{99}$$

$$-24a\eta_1^2k^2l^2\rho_1^2R(\varpi)^2-4a\eta_1^4k^2l^2R(\varpi)-4ak^2l^2\rho_1^4R(\varpi)^3-60a\eta_1^2k^2ln\rho_1^2R(\varpi)^2-10a\eta_1^4k^2lnR(\varpi)$$

$$-10ak^2ln\rho_1^4R(\varpi)^3-36a\eta_1^2k^2n^2\rho_1^2R(\varpi)^2-6a\eta_1^4k^2n^2R(\varpi)-6ak^2n^2\rho_1^4R(\varpi)^3+b_2\eta_1^6\left(-n^2\right)$$

$$-15b_2\eta_1^4n^2\rho_1^2R(\varpi)-15b_2\eta_1^2n^2\rho_1^4R(\varpi)^2-b_2n^2\rho_1^6R(\varpi)^3=0. \tag{100}$$

$$8a\eta_1^4k^2l^2\varpi+72a\eta_1^2k^2l^2\rho_1^2\varpi R(\varpi)-24a\eta_0\eta_1k^2l^2\rho_1^2R(\varpi)^2-8a\eta_0\eta_1^3k^2l^2R(\varpi)+16ak^2l^2\rho_1^4\varpi R(\varpi)^2$$

$$+18a\eta_1^4k^2ln\varpi+180a\eta_1^2k^2ln\rho_1^2\varpi R(\varpi)-72a\eta_0\eta_1k^2ln\rho_1^2R(\varpi)^2-24a\eta_0\eta_1^3k^2lnR(\varpi)$$

$$+42ak^2ln\rho_1^4\varpi R(\varpi)^2+10a\eta_1^4k^2n^2\varpi+108a\eta_1^2k^2n^2\rho_1^2\varpi R(\varpi)-48a\eta_0\eta_1k^2n^2\rho_1^2R(\varpi)^2$$

$$-16a\eta_0\eta_1^3k^2n^2R(\varpi)+26ak^2n^2\rho_1^4\varpi R(\varpi)^2+30b_2\eta_1^4n^2\rho_1^2\varpi-6b_2\eta_0\eta_1^5n^2$$

$$+60b_2\eta_1^2n^2\rho_1^4\varpi R(\varpi)-30b_2\eta_0\eta_1n^2\rho_1^4R(\varpi)^2-60b_2\eta_0\eta_1^3n^2\rho_1^2R(\varpi)+6b_2n^2\rho_1^6\varpi R(\varpi)^2=0. \tag{101}$$

$$-3n^2R(\varpi)^2b_2\rho_1^6-12n^2\varpi^2R(\varpi)b_2\rho_1^6-4ak^2l^2R(\varpi)^2\rho_1^4-10ak^2n^2R(\varpi)^2\rho_1^4-14ak^2lnR(\varpi)^2\rho_1^4$$

$$-15n^2R(\varpi)^2b_2\eta_0^2\rho_1^4-60n^2\varpi^2b_2\eta_1^2\rho_1^4-30n^2R(\varpi)b_2\eta_1^2\rho_1^4-20ak^2l^2\varpi^2R(\varpi)\rho_1^4-34ak^2n^2\varpi^2R(\varpi)\rho_1^4$$

$$-54ak^2ln\varpi^2R(\varpi)\rho_1^4-n^2R(\varpi)^2b_1\rho_1^4+120n^2\varpi R(\varpi)b_2\eta_0\eta_1\rho_1^4-15n^2b_2\eta_1^4\rho_1^2+120n^2\varpi b_2\eta_0\eta_1^3\rho_1^2$$

$$-4ak^2l^2R(\varpi)^2\eta_0^2\rho_1^2-14ak^2n^2R(\varpi)^2\eta_0^2\rho_1^2-18ak^2lnR(\varpi)^2\eta_0^2\rho_1^2-52ak^2l^2\varpi^2\eta_1^2\rho_1^2-74ak^2n^2\varpi^2\eta_1^2\rho_1^2$$

$$-126ak^2ln\varpi^2\eta_1^2\rho_1^2-90n^2R(\varpi)b_2\eta_0^2\eta_1^2\rho_1^2-24ak^2l^2R(\varpi)\eta_1^2\rho_1^2-42ak^2n^2R(\varpi)\eta_1^2\rho_1^2-66ak^2lnR(\varpi)\eta_1^2\rho_1^2$$

$$-6n^2R(\varpi)b_1\eta_1^2\rho_1^2+64ak^2l^2\varpi R(\varpi)\eta_0\eta_1\rho_1^2+134ak^2n^2\varpi R(\varpi)\eta_0\eta_1\rho_1^2+198ak^2ln\varpi R(\varpi)\eta_0\eta_1\rho_1^2$$

$$-4ak^2l^2\eta_1^4-4ak^2n^2\eta_1^4-15n^2b_2\eta_0^2\eta_1^4-8ak^2ln\eta_1^4-n^2b_1\eta_1^4+16ak^2l^2\varpi\eta_0\eta_1^3+26ak^2n^2\varpi\eta_0\eta_1^3$$

$$+42ak^2ln\varpi\eta_0\eta_1^3-4ak^2l^2R(\varpi)\eta_0^2\eta_1^2-14ak^2n^2R(\varpi)\eta_0^2\eta_1^2-18ak^2lnR(\varpi)\eta_0^2\eta_1^2=0. \tag{102}$$

$$
8n^2\varpi^3 b_2\rho_1^6+12n^2\varpi R(\varpi)b_2\rho_1^6+8ak^2l^2\varpi^3\rho_1^4+12ak^2n^2\varpi^3\rho_1^4+20ak^2ln\varpi^3\rho_1^4+60n^2\varpi R(\varpi)b_2\eta_0^2\rho_1^4
$$

$$
+60n^2\varpi b_2\eta_1^2\rho_1^4+8ak^2l^2\varpi R(\varpi)\rho_1^4+24ak^2n^2\varpi R(\varpi)\rho_1^4+32ak^2ln\varpi R(\varpi)\rho_1^4+4n^2\varpi R(\varpi)b_1\rho_1^4
$$

$$
-120n^2\varpi^2 b_2\eta_0\eta_1\rho_1^4-60n^2R(\varpi)b_2\eta_0\eta_1\rho_1^4-60n^2b_2\eta_0\eta_1^3\rho_1^2+8ak^2l^2\varpi R(\varpi)\eta_0^2\rho_1^2+34ak^2n^2\varpi R(\varpi)\eta_0^2\rho_1^2
$$

$$
+42ak^2ln\varpi R(\varpi)\eta_0^2\rho_1^2+180n^2\varpi b_2\eta_0^2\eta_1^2\rho_1^2+32ak^2l^2\varpi\eta_1^2\rho_1^2+52ak^2n^2\varpi\eta_1^2\rho_1^2+84ak^2ln\varpi\eta_1^2\rho_1^2
$$

$$
+12n^2\varpi b_1\eta_1^2\rho_1^2-60n^2R(\varpi)b_2\eta_0^3\eta_1\rho_1^2-40ak^2l^2\varpi^2\eta_0\eta_1\rho_1^2-80ak^2n^2\varpi^2\eta_0\eta_1\rho_1^2-120ak^2ln\varpi^2\eta_0\eta_1\rho_1^2
$$

$$
-16ak^2l^2R(\varpi)\eta_0\eta_1\rho_1^2-50ak^2n^2R(\varpi)\eta_0\eta_1\rho_1^2-66ak^2lnR(\varpi)\eta_0\eta_1\rho_1^2-12n^2R(\varpi)b_1\eta_0\eta_1\rho_1^2-20n^2b_2\eta_0^3\eta_1^3
$$

$$
-8ak^2l^2\eta_0\eta_1^3-10ak^2n^2\eta_0\eta_1^3-18ak^2ln\eta_0\eta_1^3-4n^2b_1\eta_0\eta_1^3+8ak^2l^2\varpi\eta_0^2\eta_1^2+22ak^2n^2\varpi\eta_0^2\eta_1^2
$$

$$
+30ak^2ln\varpi\eta_0^2\eta_1^2-4ak^2n^2R(\varpi)\eta_0^3\eta_1-4ak^2lnR(\varpi)\eta_0^3\eta_1=0. \tag{103}
$$

$$
-4a\eta_0^2k^2l^2\rho_1^2\varpi^2+16a\eta_0\eta_1k^2l^2\rho_1^2\varpi-4a\eta_1^2k^2l^2\rho_1^2-4a\eta_0^2\eta_1^2k^2l^2-4ak^2l^2\rho_1^4\varpi^2-18a\eta_0^2k^2ln\rho_1^2\varpi^2
$$

$$
+66a\eta_0\eta_1k^2ln\rho_1^2\varpi-12a\eta_1^2k^2ln\rho_1^2+6a\eta_0^3\eta_1k^2ln\varpi-12a\eta_0^2\eta_1^2k^2ln-14ak^2ln\rho_1^4\varpi^2-12a\eta_0^2k^2ln\rho_1^2R(\varpi)
$$

$$
-4ak^2ln\rho_1^4R(\varpi)-14a\eta_0^2k^2n^2\rho_1^2\varpi^2+50a\eta_0\eta_1k^2n^2\rho_1^2\varpi-8a\eta_1^2k^2n^2\rho_1^2+6a\eta_0^3\eta_1k^2n^2\varpi-8a\eta_0^2\eta_1^2k^2n^2
$$

$$
-10ak^2n^2\rho_1^4\varpi^2-12a\eta_0^2k^2n^2\rho_1^2R(\varpi)-4ak^2n^2\rho_1^4R(\varpi)-60b_2\eta_0^2n^2\rho_1^4\varpi^2+120b_2\eta_0\eta_1n^2\rho_1^4\varpi
$$

$$
+120b_2\eta_0^3\eta_1n^2\rho_1^2\varpi+24b_1\eta_0\eta_1n^2\rho_1^2\varpi-15b_2\eta_1^2n^2\rho_1^4-90b_2\eta_0^2\eta_1^2n^2\rho_1^2-6b_1\eta_1^2n^2\rho_1^2-15b_2\eta_0^4\eta_1^2n^2
$$

$$
-6b_1\eta_0^2\eta_1^2n^2-12b_2n^2\rho_1^6\varpi^2-4b_1n^2\rho_1^4\varpi^2-30b_2\eta_0^2n^2\rho_1^4R(\varpi)-15b_2\eta_0^4n^2\rho_1^2R(\varpi)-6b_1\eta_0^2n^2\rho_1^2R(\varpi)
$$

$$
-3b_2n^2\rho_1^6R(\varpi)-2b_1n^2\rho_1^4R(\varpi)+\eta_1^2ln^2\omega+ln^2\rho_1^2\omega R(\varpi)=0. \tag{104}
$$

$$
6a\eta_0^2k^2ln\rho_1^2\varpi-6a\eta_0\eta_1k^2ln\rho_1^2-2a\eta_0^3\eta_1k^2ln+2ak^2ln\rho_1^4\varpi+6a\eta_0^2k^2n^2\rho_1^2\varpi-6a\eta_0\eta_1k^2n^2\rho_1^2
$$

$$
-2a\eta_0^3\eta_1k^2n^2+2ak^2n^2\rho_1^4\varpi+60b_2\eta_0^2n^2\rho_1^4\varpi+30b_2\eta_0^4n^2\rho_1^2\varpi+12b_1\eta_0^2n^2\rho_1^2\varpi-30b_2\eta_0\eta_1n^2\rho_1^4
$$

$$
-60b_2\eta_0^3\eta_1n^2\rho_1^2-12b_1\eta_0\eta_1n^2\rho_1^2-6b_2\eta_0^5\eta_1n^2-4b_1\eta_0^3\eta_1n^2+6b_2n^2\rho_1^6\varpi+4b_1n^2\rho_1^4\varpi
$$

$$
+2\eta_0\eta_1ln^2\omega-2ln^2\rho_1^2\varpi\omega=0. \tag{105}
$$

$$-15b_2\eta_0^4n^2\rho_1^2-15b_2\eta_0^2n^2\rho_1^4-6b_1\eta_0^2n^2\rho_1^2+b_2\eta_0^6\left(-n^2\right)-b_1\eta_0^4n^2-b_2n^2\rho_1^6$$

$$-b_1n^2\rho_1^4+\eta_0^2ln^2\omega+ln^2\rho_1^2\omega=0. \tag{106}$$

The simultaneous computation of these equations produces the following results:
Case 1 $R(\varpi)=0$.

$$k=k=\pm n\sqrt{\frac{b_1}{2al^2+4aln+2an^2}},\ \eta_0=0,\ \eta_1=0,\ \rho_1=\pm\sqrt{\frac{-2b_1l-3b_1n}{4\left(b_2l+b_2n\right)}},\ \omega=-\frac{b_1^2\left(4l^2+8ln+3n^2\right)}{16b_2l(l+n)^2}. \tag{107}$$

The solution to the governing problem Eq. (68) is obtained with the conditions $\frac{b_1(2l+3n)}{4b_2l+4b_2n}<0$ and $\frac{b_1}{(al^2+2aln+an^2)}>0$:

$$q(x,\,t)=\left(\sqrt{-\frac{b_1(2l+3n)}{4b_2l+4b_2n}}\tanh\left(n\sqrt{\frac{b_1}{8\left(al^2+2aln+an^2\right)}}x\right)\right)^{\frac{2}{n}}e^{i\left(-\frac{b_1^2\left(4l^2+8ln+3n^2\right)}{16b_2l(l+n)^2}t+\theta\right)}, \tag{108}$$

or

$$q(x,\,t)=\left(\sqrt{-\frac{b_1(2l+3n)}{4b_2l+4b_2n}}\coth\left(n\sqrt{\frac{b_1}{8\left(al^2+2aln+an^2\right)}}x\right)\right)^{\frac{2}{n}}e^{i\left(-\frac{b_1^2\left(4l^2+8ln+3n^2\right)}{16b_2l(l+n)^2}t+\theta\right)}. \tag{109}$$

Case 2 $R(\varpi)=\frac{24}{25}\varpi^2$.

$$k=\pm n\sqrt{\frac{b_1(2l+3n)}{2(l+n)\left(2al^2+5aln+3an^2\right)}},\ \eta_0=0,\ \eta_1=\pm\varpi\sqrt{\frac{6b_1(2l+3n)}{25\left(b_2(-l)-b_2n\right)}},$$

$$\rho_1=\pm\sqrt{\frac{b_1(2l+3n)}{4\left(b_2(-l)-b_2n\right)}},\ \omega=-\frac{b_1^2(2l+n)(2l+3n)}{16b_2l(l+n)^2}. \tag{110}$$

Consequently, the closed-form solution to the governing model Eq. (68) for $\frac{b_1(2l+3n)}{4b_2l+4b_2n}<0$ and $\frac{b_1(2l+3n)}{2(l+n)(2al^2+5aln+3an^2)}>0$ is attained:

$$q(x,t)=\left\{\pm\sqrt{\frac{b_1(2l+3n)}{b_2(-l)-b_2n}}\left(\frac{\sqrt{6}\operatorname{sech}\left(n\sqrt{\frac{b_1(2l+3n)}{2(l+n)(2al^2+5aln+3an^2)}}x\right)}{5\operatorname{sech}\left(n\sqrt{\frac{b_1(2l+3n)}{2(l+n)(2al^2+5aln+3an^2)}}x\right)\pm 1}\right.\right.$$

$$\left.\left.\pm\frac{\tanh\left(n\sqrt{\frac{b_1(2l+3n)}{2(l+n)(2al^2+5aln+3an^2)}}x\right)}{2\left(1\pm 5\operatorname{sech}\left(n\sqrt{\frac{b_1(2l+3n)}{2(l+n)(2al^2+5aln+3an^2)}}x\right)\right)}\right)\right\}^{\frac{2}{n}} e^{i\left(-\frac{b_1^2(2l+n)(2l+3n)}{16b_2l(l+n)^2}t+\theta\right)}. \tag{111}$$

Case 3 $R(\varpi)=\frac{5}{9}\varpi^2$.

$$k=\pm n\sqrt{\frac{b_1(2l+3n)}{2(-l-n)(-2al^2-5aln-3an^2)}},\ \eta_0=0,\ \eta_1=\pm\varpi\sqrt{\frac{5b_1(2l+3n)}{36(b_2(-l)-b_2n)}},$$

$$\rho_1=\pm\sqrt{\frac{b_1(2l+3n)}{4(b_2(-l)-b_2n)}},\ \omega=-\frac{b_1^2(2l+n)(2l+3n)}{16b_2l(l+n)^2}. \tag{112}$$

Consequently, the closed-form solution to the governing model Eq. (68) for $\frac{b_1(2l+3n)}{b_2l+b_2n}<0$ and $\frac{b_1(2l+3n)}{2(-l-n)(-2al^2-5aln-3an^2)}>0$ is attained:

$$q(x,t)$$

$$=\left\{\pm\sqrt{\frac{b_1(2l+3n)}{-b_2l-b_2n}}\left(\frac{\sqrt{5}\operatorname{sech}\left(n\sqrt{\frac{b_1(2l+3n)}{2(-l-n)(-2al^2-5aln-3an^2)}}x\right)}{2\left(3\operatorname{sech}\left(n\sqrt{\frac{b_1(2l+3n)}{2(-l-n)(-2al^2-5aln-3an^2)}}x\right)\pm 2\right)}\right.\right.$$

$$\left.\left.+\frac{1}{3\operatorname{csch}\left(n\sqrt{\frac{b_1(2l+3n)}{2(-l-n)(-2al^2-5aln-3an^2)}}x\right)+(\pm 2)\coth\left(n\sqrt{\frac{b_1(2l+3n)}{2(-l-n)(-2al^2-5aln-3an^2)}}x\right)}\right)\right\}^{\frac{2}{n}}$$

$$\times e^{i\left(-\frac{b_1^2(2l+n)(2l+3n)}{16b_2l(l+n)^2}t+\theta\right)}. \tag{113}$$

Case 4 $R(\varpi) = \varpi^2 - 1$.

$$k = \pm n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}},\ \eta_0 = 0,\ \eta_1 = \pm\sqrt{\frac{(\varpi^2 - 1)(-2b_1 l - 3b_1 n)}{4(b_2 l + b_2 n)}}, \tag{114}$$

$$\rho_1 = \pm\sqrt{\frac{-2b_1 l - 3b_1 n}{4(b_2 l + b_2 n)}},\ \omega = -\frac{b_1^2(2l+n)(2l+3n)}{16b_2 l(l+n)^2}.$$

Consequently, the closed-form solution to the governing model Eq. (68) for $\frac{b_1(2l+3n)}{b_2 l + b_2 n} < 0$ and $\frac{b_1}{2al^2 + 4aln + 2an^2} > 0$ is attained:

$$\begin{aligned} &q(x, t) \\ &= \left\{ \pm\frac{1}{2}\sqrt{\frac{-2b_1 l - 3b_1 n}{b_2 l + b_2 n}} \left(\frac{4\sqrt{\varpi^2 - 1}\operatorname{sech}\left(n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}}x\right)}{\left(4\varpi\operatorname{sech}\left(n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}}x\right) \pm 3\tanh\left(n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}}x\right)\right) + 5} \right.\right. \\ &\left.\left. + \frac{3 \pm 5\tanh\left(n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}}x\right)}{\left(4\varpi\operatorname{sech}\left(n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}}x\right) \pm 3\tanh\left(n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}}x\right)\right) + 5} \right) \right\}^{\frac{2}{n}} \\ &\times e^{i\left(-\frac{b_1^2(2l+n)(2l+3n)}{16b_2 l(l+n)^2}t + \theta\right)}. \end{aligned} \tag{115}$$

or

$$\begin{aligned} q(x, t) = &\left\{ \pm\frac{1}{2}\sqrt{\frac{-2b_1 l - 3b_1 n}{b_2 l + b_2 n}} \left(\frac{\sqrt{\varpi^2 - 1}\operatorname{sech}\left(n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}}x\right)}{\varpi\operatorname{sech}\left(n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}}x\right) + 1} \right.\right. \\ &\left.\left. \pm \frac{\tanh\left(n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}}x\right)}{\varpi\operatorname{sech}\left(n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}}x\right) + 1} \right) \right\}^{\frac{2}{n}} e^{i\left(-\frac{b_1^2(2l+n)(2l+3n)}{16b_2 l(l+n)^2}t + \theta\right)}. \end{aligned} \tag{116}$$

Case 5 $R(\varpi) = \varpi^2 + 1$.

$$k = \pm n\sqrt{\frac{b_1}{2al^2 + 4aln + 2an^2}},\ \eta_0 = 0,\ \eta_1 = \pm\sqrt{\frac{(\varpi^2+1)(-2b_1 l - 3b_1 n)}{4(b_2 l + b_2 n)}},$$

$$\rho_1 = \pm\sqrt{\frac{-2b_1 l - 3b_1 n}{4(b_2 l + b_2 n)}},\ \omega = -\frac{b_1^2(2l+n)(2l+3n)}{16b_2 l(l+n)^2}. \tag{117}$$

Consequently, the closed-form solution to the governing model Eq. (68) for $\frac{b_1(2l+3n)}{b_2 l + b_2 n} < 0$ and $\frac{b_1}{2al^2 + 4aln + 2an^2} > 0$ is attained:

$$q(x, t)\left\{\pm\frac{1}{2}\sqrt{\frac{-2b_1 l - 3b_1 n}{b_2 l + b_2 n}}\left(\frac{\coth\left(n\sqrt{\frac{b_1}{2al^2+4aln+2an^2}}x\right)}{1 \pm \varpi \operatorname{csch}\left(n\sqrt{\frac{b_1}{2al^2+4aln+2an^2}}x\right)}\right.\right.$$

$$\left.\left.+\frac{\sqrt{\varpi^2+1}\operatorname{csch}\left(n\sqrt{\frac{b_1}{2al^2+4aln+2an^2}}x\right)}{1 \pm \varpi \operatorname{csch}\left(n\sqrt{\frac{b_1}{2al^2+4aln+2an^2}}x\right)}\right)\right\}^{\frac{2}{n}} e^{i\left(-\frac{b_1^2(2l+n)(2l+3n)}{16b_2 l(l+n)^2}t + \theta\right)}. \tag{118}$$

5. Results and discussion

This section provides a detailed analysis of the wave profiles of quiescent dark and singular solitons under various nonlinear and dispersive effects. Specifically, the influences of power-law nonlinearity, nonlinear CD, quadratic and cubic nonlinearities, and generalized temporal evolution are examined. These effects are illustrated through Figures 1-4, using parameter values $a = 1$, $b_1 = 1$, and $b_2 = -1$.

Figure 1 illustrates the wave profile of the quiescent dark soliton (41) and its behavior under various nonlinear and dispersive influences. In Figure 1a, the power-law nonlinearity significantly alters the soliton's amplitude and width. As the nonlinearity exponent increases, the soliton profile exhibits sharper dips with enhanced contrast against the background. This suggests that stronger power-law effects deepen the soliton's intensity, making it more localized. In Figure 1b, nonlinear CD modifies the soliton shape by introducing asymmetric broadening or compression. Depending on the sign and magnitude of the dispersion coefficient, the dark soliton either stretches or contracts, affecting its stability and propagation characteristics. In Figure 1c, quadratic nonlinearity influences the soliton's depth and symmetry. The presence of this nonlinearity leads to a shift in the soliton core, modifying its contrast. It also plays a crucial role in governing energy redistribution within the system, impacting soliton stability. In Figure 1d, the cubic nonlinear effect further refines the soliton structure, leading to sharper gradients and a more confined shape. This nonlinear contribution stabilizes the dark soliton, preventing excessive broadening and maintaining its integrity during propagation.

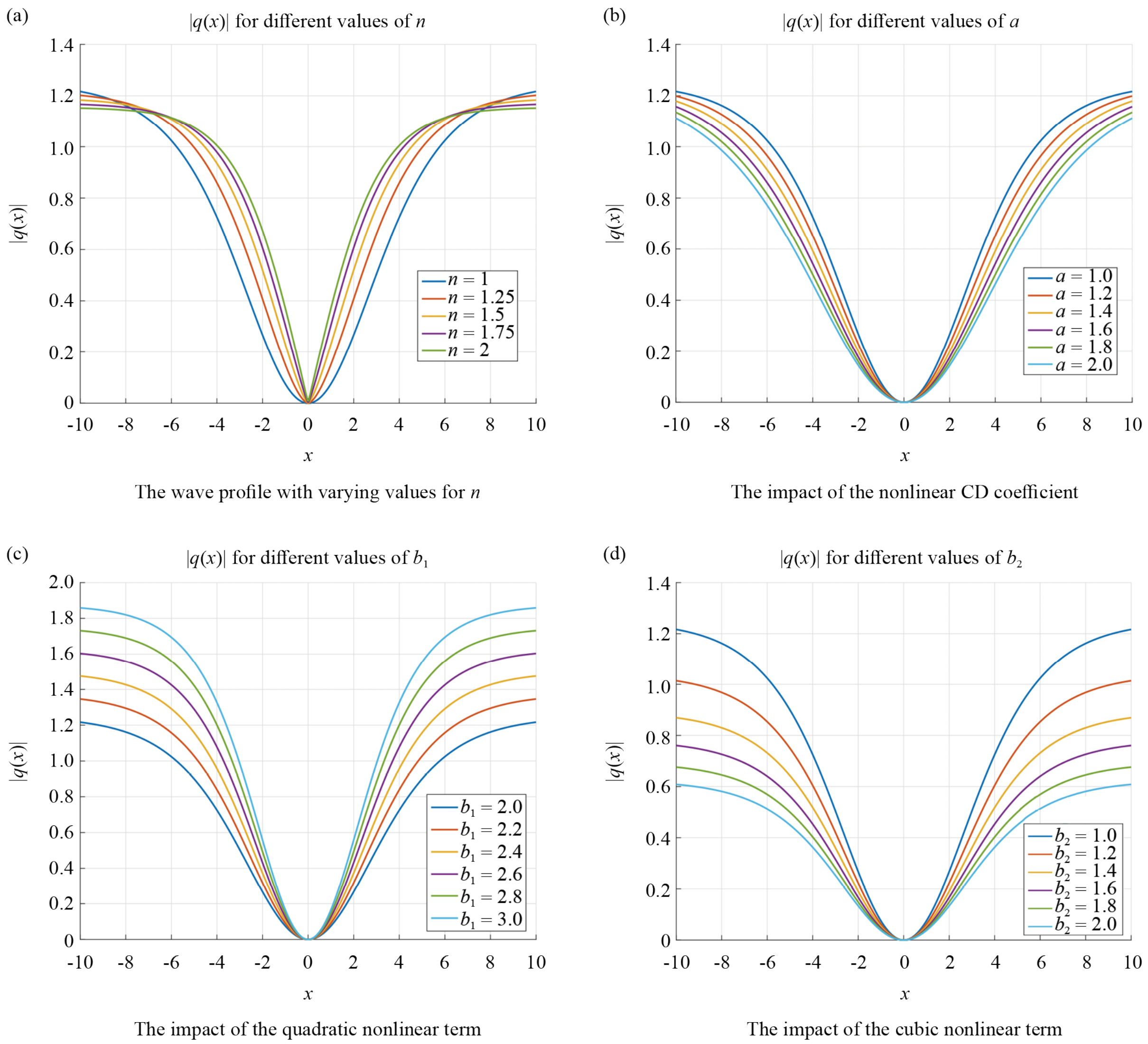

Figure 1. Wave profile of the quiescent dark soliton (41) under various impacts

Figure 2 examines the wave profile of the quiescent singular soliton (42) and its response to different nonlinear and dispersive effects. In Figure 2a, singular solitons are highly sensitive to power-law nonlinearity. As the nonlinearity parameter increases, the soliton peak intensifies, and its steepness becomes more pronounced. This suggests that power-law effects enhance singularity formation, reinforcing the soliton's self-trapping ability. In Figure 2b, the inclusion of nonlinear CD distorts the singular soliton profile, either enhancing its peak intensity or broadening its base. This dispersion-induced modification significantly affects soliton dynamics, potentially leading to bifurcation or mode transitions. In Figure 2c, quadratic nonlinearity introduces a secondary modulation effect, which can lead to soliton splitting or deformation. Depending on the balance between nonlinearity and dispersion, the singular soliton profile either stabilizes or experiences structural instability. In Figure 2d, the cubic nonlinearity refines the singular soliton peak, enhancing its sharpness and confinement. This stabilizing effect ensures that the soliton remains localized despite external perturbations, highlighting the cubic nonlinearity's crucial role in soliton maintenance.

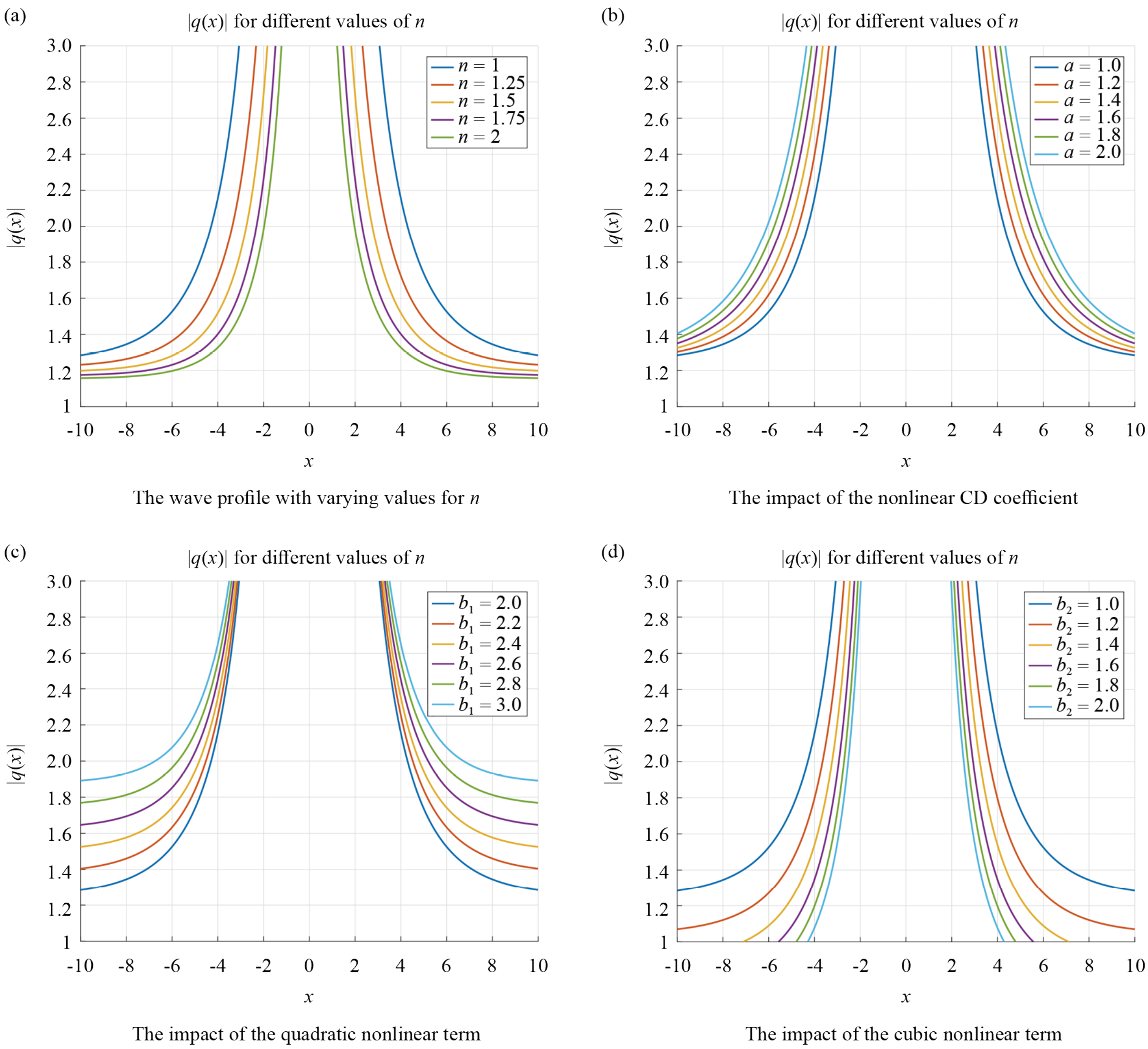

Figure 2. Wave profile of the quiescent singular soliton (42) under various impacts

Figure 3 focuses on the wave profile of the quiescent dark soliton (91) and its evolution under generalized temporal effects and power-law nonlinearity. In Figure 3a, the generalized temporal evolution introduces a time-dependent modulation, leading to dynamic shape alterations in the dark soliton. This effect results in gradual intensity variations, influencing soliton breathing modes and stability thresholds. In Figure 3b, as seen in previous cases, power-law nonlinearity enhances the depth and localization of the dark soliton. The balance between temporal evolution and power-law effects dictates whether the soliton remains stable or undergoes distortion.

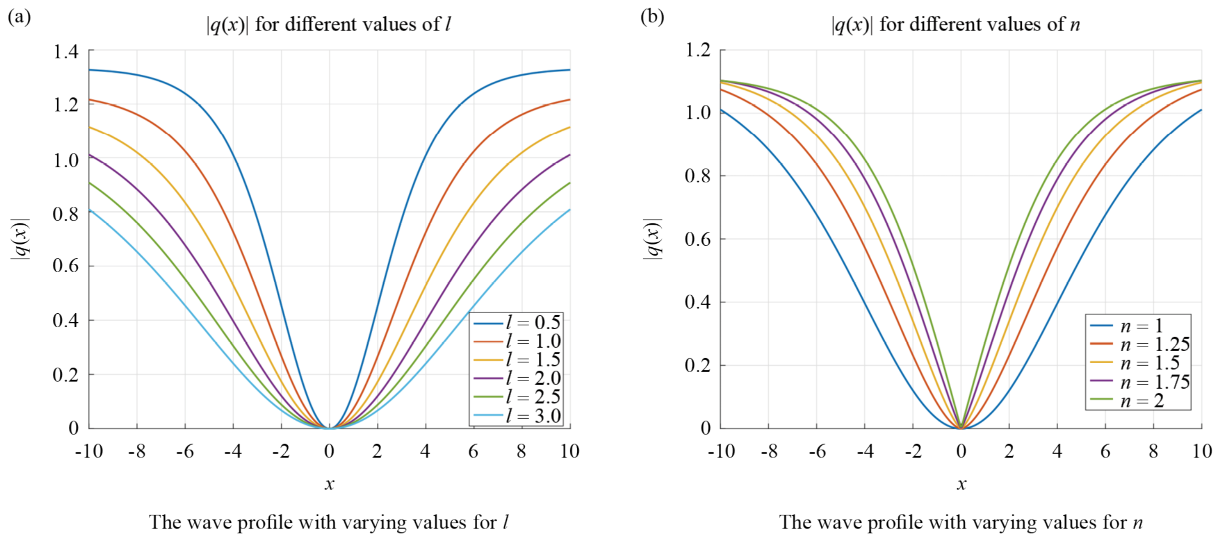

Figure 3. Wave profile of the quiescent dark soliton (91) with generalized temporal evolution

Figure 4 examines the quiescent singular soliton (92) and its behavior under generalized temporal evolution and power-law nonlinearity. In Figure 4a, temporal evolution significantly impacts singular soliton behavior, leading to self-modulation effects. These effects can cause oscillatory dynamics or rapid peak variations, influencing soliton longevity and robustness. In Figure 4b, the power-law nonlinearity reinforces the singular soliton's peak intensity, further sharpening its singular characteristics. This interaction dictates the soliton's ability to resist perturbations and maintain structural coherence.

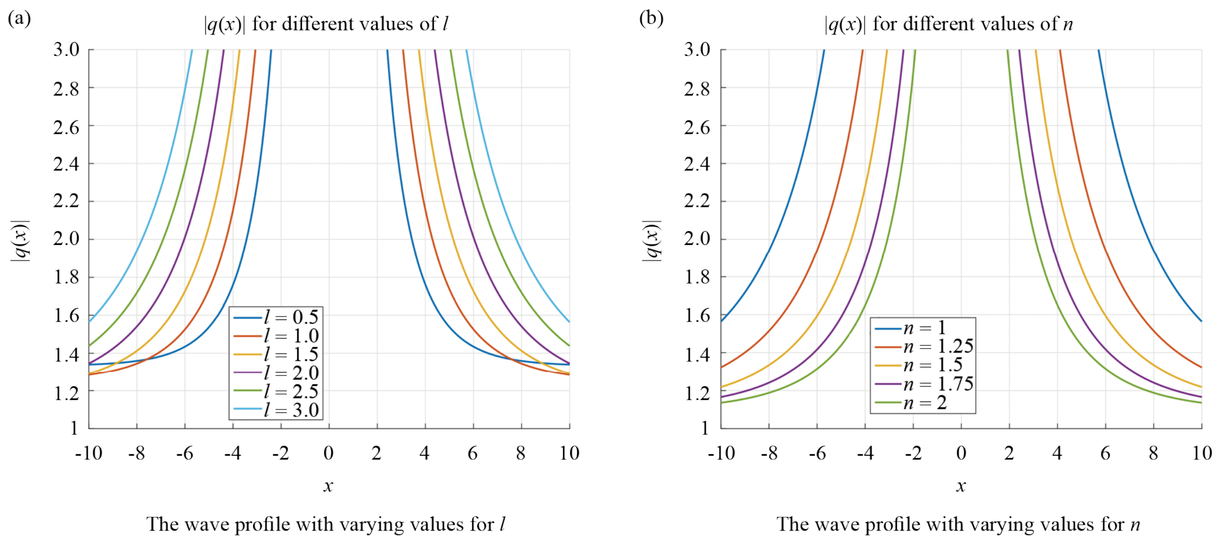

Figure 4. Wave profile of the quiescent singular soliton (92) with generalized temporal evolution

The results indicate that power-law nonlinearity, nonlinear CD, quadratic and cubic nonlinearities, and generalized temporal evolution play critical roles in shaping soliton dynamics. Dark solitons exhibit deepening and sharpening effects under increasing nonlinearity, whereas singular solitons experience enhanced confinement and self-trapping. Nonlinear CD and generalized temporal evolution introduce dynamic modulations, affecting soliton stability and propagation properties. These findings provide valuable insights into the control and manipulation of quiescent solitons in nonlinear optical systems.

6. Conclusions

This paper addressed the perturbed Fokas-Lenells equation with nonlinear CD with the application of two integration schemes to recover quiescent optical solitons. The temporal evolutions were both linear and generalized. The enhanced Kudryashov's approach and the projective Riccati equation scheme yielded the quiescent solitons that are listed in the current work. The analytical results are supplemented with numerical simulations. The results are indeed very promising to move on with additional forms of SPM and to additional forms of optoelectronic devices. A few such applications would be with polarization-mode dispersion and dispersion-flattened fibers. Later, the model would be applied to study the evolution of such quiescent solitons in magneto-optic waveguides, optical metamaterials, optical couplers and other such devices. The results would be recovered and aligned with the pore-existing ones before they are disseminated across the board in various outlets [4–7].

Acknowledgement

One of the authors, AB, is grateful to Grambling State University for the financial support he received as the Endowed Chair of Mathematics. This support is sincerely appreciated.

Conflict of interest

The authors claim that there is no conflict of interest.

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