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Research Article

Mathematical Evaluation and Dynamic Transmissions of a Tumor Growth Model Using a Generalized Singular and Non-Local Kernel

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Abstract: Currently, fractional calculus plays a critical role in improving control techniques, analyzing disease transmission dynamics, and solving several other real-world problems. This research investigates the time-fractional tumor growth model using an innovative approach. The new modified fractional derivative operator employs a singular and non-local kernel, based on Atangana and Baleanu's concepts with the Caputo derivative. The tumor growth model used the newly modified fractional operator, which provided numerical simulation. With the introduction of this new operator, we provide significant analysis for the tumor growth epidemic model. We have proven the uniqueness and stability conditions of the model by utilizing Banach's fixed point theory and the Picard successive approximation method. Using the Laplace-Adomian decomposition method (LADM), we found the numerical solution to the Modified Atangana-Baleanu-Caputo derivative model. We have verified the convergence analysis of the suggested scheme. We ultimately utilize the suggested method to obtain numeric outcomes and simulations for the tumor growth model. The study investigates the effect of multiple biological variables on the transmission of tumor growth dynamics.

***Keywords*:** tumor growth model, ladm, picard iteration method, banach's fixed-point theory, mabc derivative

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1. Introduction

Tumors are classified into two categories: malignant tumors, or cancerous tumors, and benign tumors, or non-cancerous tumors, are the two categories into which tumors fall. A tumor refers to an abnormal growth of bodily tissue. Despite the progress made in the fields of science and medicine, tumors continue to be a prevalent and sometimes fatal disease. A tumor is the most deadly and complex disease of our time. It is a multi-stage disease that develops from abnormal cells' altered DNA creation, or mutation. Tumors are the result of uncontrolled cell growth and frequent genetic alterations within the human organism. While a portion of these modified cells perish, some survive and develop into malignant cells. Tumors can arise when the other protective systems or immune systems in the human body are unable to defend against these cells [1].

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The primary challenge in disease transmission is to efficiently analyze and regulate the epidemic's pattern, utilizing existing information and a mathematical model. A precise mathematical model should take into account past, present, and future aspects of the disease. Fractional calculus (FC) is a mathematical framework that generalizes integral operations and derivatives to non-integer orders, allowing for the representation of long-term memory. FC has gained significant popularity in various disciplines, such as biology, engineering, and physics [2]. Researchers have discovered coping techniques in FC formulation, such as vaccinations, therapy, and isolation [3]. To investigate the fractional optimum control problems related to various diseases, we can select the suitable fractional order and fractional operator by keeping the control variable at a consistent level [4, 5]. Various types of fractional differential operators exist. The most common are Caputo, which is singular and nonlocal and focuses on the power law; Caputo-Fabrizio, which is nonsingular and focuses on the exponential decay law; and Atangana-Baleanu, which is nonsingular and nonlocal and focuses on the generalized Mittag-Leffler function. Recently [6, 7], were able to solve the issue the operator described in [8]. In order to tackle the aforementioned concerns regarding operators that are not singular, the authors in [9] made adjustments to the operator by incorporating a Mittag-Leffler kernel. They demonstrated that the resulting fractional differential equations (FDEs) associated with this modified operator are easy to initialize. Moreover, they established that the MABC derivative can effectively resolve a number of unsolvable fractional differential problems using the ABC derivative. At the origin, the MABC derivative displays an integrable singularity. The researchers introduced an innovative numerical technique based on finite differences in [10] for computing the MABC derivative. This technique greatly simplifies the initialization of the corresponding fractional differential equations. Researchers utilized the MABC derivative, including Mittag-Leffler kernel, to construct modified fractional difference operators [11]. The modified Atangana-Baleanu-Caputo fractional derivative is a valuable tool for accurately simulating the dynamics of disease. It provides a detailed depiction of the complicated biological processes involved, thanks to its singular and non-local kernel. This approach not only improves our understanding of tumor biology but also provides a solid foundation for investigating therapy methods and improving treatment methods. It is thought that the Modified Atangana-Baleanu-Caputo fractional derivative was chosen to show how tumors grow because it can show complex biological interactions, allow for abnormal spreading effects, and provide a flexible framework for studying how tumor growth methods depend on time and memory effects. These benefits make it a convincing option compared to current methodologies, especially when striving for a more precise and authentic presentation of biological processes. Using these forms of kernels, modeling and simulation are greatly impacted by the area of fractional order differential and integrals equations [12–16]. It has been observed that models utilizing differential equations and fractional order integrals exhibit higher levels of accuracy compared to classical models [17–20]. Each of these differential operators, which are connected in some way, has unique characteristics of its own. Depending on the aspect from which the question is being researched, different relevant representations are typically provided. In mathematical models, especially epidemic models, the fractional order operators are considerably more beneficial in assisting individuals in comprehending real-world phenomena. Moreover, these models have the ability to more accurately represent inversion behavior and dissolving memory. Further details about a disease are provided by mathematical models with fractional derivatives.

Integral transformations are essential tools in mathematics and applied sciences, serving as effective methods for solving and analyzing both integral and differential equations, as well as helping to gain a better understanding of the behavior of complicated systems. Here are some integral transforms: Laplace transform, Hilbert transform, Fourier transform, and Mellin transform. Integral transformations have diverse applications in engineering, finance, physics, biology, and other domains, showcasing their diversity and usefulness. In many instances, transforms result in more efficient processing techniques than direct numerical differentiation or integration [21]. In LADM, the Laplace transform and the Adomian decomposition method works well together to handle the numerical solution to the modified Atangana-Baleanu-Caputo derivative model. By putting the problem in the Laplace domain and using the Adomian decomposition method, it gives a methodical way to solve fractional differential equations that are challenging to solve analytically. The Picard iterative method and Banach's fixed-point theorem are powerful mathematical tools for demonstrating that solutions exist, are singular, and are stable in the structure of the tumor growth epidemic model. These methods provide both theoretical exactness and practical application in the fields of biological modeling and epidemiological studies.

The research by Chavada et al. [22] showed that the breast cancer model was correct by using the Caputo Fabrizio fractional derivative, Picard-Lindelof (PL) method, and Ulam Hyers (UH) criteria. The fractional two step Adams-Bashforth approach has been employed to obtain numeric outcomes. In their study, Shah et al. [23] introduced a smoking tobacco cancer model utilizing the Caputo fractional operator. The reproduction number idea is utilized to analyze the stability of the spread of cancer caused by tobacco. The breast cancer competition model was introduced by Solís-Pérez et al. [24] incorporating the Caputo-Fabrizio and Caputo fractional derivatives. Using the Laplace transform, unique solutions were discovered, and the Atangana-Toufik numerical technique produced numerical simulations of the model for both derivatives. A Caputo derivative fractional cancer tumor model was proposed by Attia Nourhane et al. [25] and subsequently investigated using the reproducing kernel Hilbert space method. A discrete fractional tumor-immune model under the Caputo operator was discussed by Alzabut Jehad et al. [26]. This model has been proven through Leray-Schauder's and Banach's fixed point theorems. Amilo et al. [27] present a fractional-order lung cancer model, describing its exploration of the effects of surgery and immunotherapy on tumor growth rate and the immune system's response to cancer cells.

The proposed model, which examines the influence of irradiation on tumor growth, employs a fractional-order model with a singular and non-local kernel. This model has potential applications that extend beyond the dynamics of tumor growth. To investigate drug delivery systems within the human body, we can use the fractional-order model. These models may analyze the dispersion and interaction of medications with tissues over a period of time through the process of modeling. This analysis can be used to improve drug dosing techniques and precisely predict drug efficacy under a variety of physiological conditions. This research has the potential to significantly impact tumor treatment techniques and advance our knowledge of tumor biology.

By exploring the research, one can find numerous cancer models that cover different perspectives on tumors. In their study, Watanabe et al. [28] presented a tumor growth model that describes the changes in tumor volume over time, both prior to and following radiosurgical procedures. This work focuses on a model that considers the tumor's growth following a single instant irradiation dose delivered to the tumor at a given time t. We employ the tumor growth model that was provided by

$$\begin{aligned} \frac{d\mathbf{u}(t)}{dt} &= \mathbf{w}(t)q(\mathscr{D})\mathbf{u}(t) - h(\mathscr{D})\mathbf{u}(t), \\ \frac{d\mathbf{v}(t)}{dt} &= h(\mathscr{D})\mathbf{u}(t) - \mu_c\mathbf{v}(t), \\ \frac{d\mathbf{w}(t)}{dt} &= -\varpi\mathbf{w}(0)\mathbf{w}(t). \end{aligned} \tag{1}$$

Parameters and description are given by Table 1.Diagram for model of tumor growth is given by Figure 1. The following represents the way the manuscript is organized: The section "Introduction" contains the manuscript's introduction and motivation. Fundamental definitions of fractional calculus and Laplace transform are addressed in the section under "Basic concepts." In the third section, the construction of fractional order models and proven lemmas for the suggested model are discussed. The fourth section describes an iterative approach for solving the model indicated previously using the LADM. Furthermore, "Picard successive approximation process" and "Banach fixed point theory" are employed to analyze stability criteria. The analysis and graphical representation of the numerical simulations for different fractional-order values are presented in Section Five of the paper. The concept of convergence is discussed in Section six, followed by the presentation of our conclusion in Section seven.

Table 1. Parameters and description

Parameters	Description
$\mathscr{D}$	Rate of radiation dosage
$\mathbf{u}(t)$	Volume of proliferating tumor
$\mathbf{v}(t)$	Volume composed of non-dividing cells
$\mathbf{w}(t)$	Tumor growth rate
$\mathbf{w}(0)$	Initial tumor growth rate
μ_c	Cell clearance rate
ϖ	Vascular growth retardation
$q(\mathscr{D})$	Cell proliferation probability
$h(\mathscr{D})$	Non-dividing state with a rate

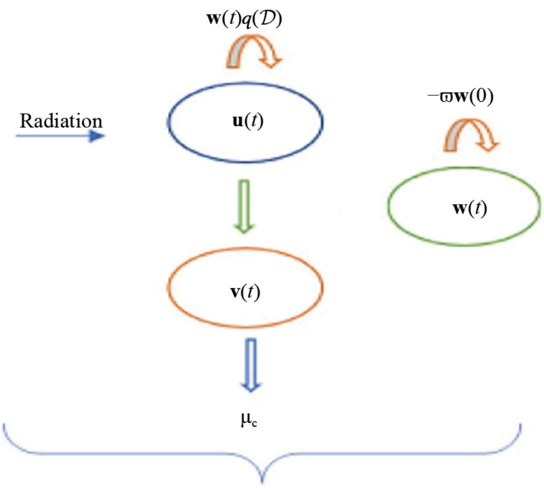

Figure 1. Diagram for model of tumor growth

2. Basic concepts

This section provides fundamental definitions derived via fractional calculus.

Definition 1 [7] Let $\Upsilon \in H^1([0,\ T])$, $T > 0$, and $\alpha \in (0,\ 1)$. For a function $\Upsilon(t)$ of order α, the Riemann-Liouville time-fractional derivative given as

$$^{\mathrm{RL}}D_t^{\alpha}\Upsilon(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\varphi)^{-\alpha}\Upsilon(\varphi)d\varphi, \tag{2}$$

the Riemann-Liouville derivative's Laplace transform is given by

$$\mathscr{L}\left\{^{\mathrm{RL}}D_t^{\alpha}\Upsilon(t)\right\} = p^{\alpha}\mathscr{L}\left\{\Upsilon(t) - {}_t^{\alpha-1}\Upsilon(t)\right\}\big|_{t=0}.$$

Definition 2 [7] Let $\Upsilon \in H^1([0,\ T])$, $T > 0$, and $\alpha \in (0,\ 1)$. For a function $\Upsilon(t)$ of order α, the Caputo timefractional derivative is given as

$$ {}^{\mathrm{C}}D_t^{\alpha}\Upsilon(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\varphi)^{-\alpha} \Upsilon'(\varphi) d\varphi. \tag{3} $$

the Caputo derivative's Laplace transform is given by

$$ \mathscr{L}\{{}^{\mathrm{c}}D_t^{\alpha}\Upsilon(t)\} = p^{\alpha}\mathscr{L}\{\Upsilon(t)\} - p^{\alpha-1}\Upsilon(0). $$

Definition 3 [8] Let $\Upsilon \in H^1([0,\ T]),\ T > 0$, and $\alpha \in (0,\ 1)$. For a function $\Upsilon(t)$ of order α, the Caputo-Fabrizio derivative is given as

$$ {}^{\mathrm{CF}}_{\ 0}D_t^{\alpha}\Upsilon(t) = \frac{\mathrm{M}(\alpha)}{1-\alpha} \int_0^t \frac{d}{dt}\Upsilon(\varphi) \exp\left(-\frac{\alpha(t-\varphi)}{(1-\alpha)}\right) d\varphi. \tag{4} $$

where $\mathrm{M}(\alpha)$ is a normalization function and $\mathrm{M}(0) = \mathrm{M}(1) = 1$.

The Caputo-Fabrizio derivative's Laplace transform is given by

$$ \mathscr{L}\left\{{}^{\mathrm{CF}}D_t^{\alpha}\Upsilon(t)\right\} = \frac{p\mathscr{L}(Y(t)\} - \Upsilon(0)}{(1-\alpha)p+\alpha}, \left|\frac{\alpha/(1-\alpha)}{p}\right| < 1. $$

Definition 4 [9] Let $\Upsilon \in \mathrm{L}^1([0,\ \mathrm{T}])$, the MABC derivative with order $0 < \alpha < 1$, is given as follows:

$$ {}^{\mathrm{MABC}}_{\quad\ 0}D_t^{\alpha}\Upsilon(t) = \frac{\mathrm{AB}(\alpha)}{1-\alpha}\left[\Upsilon(t) - \mathrm{E}_{\alpha}\left(-\eta_{\alpha}t^{\alpha}\right)\Upsilon(0)\right] - \eta_{\alpha}\int_0^t (t-\varphi)^{\alpha-1}\mathrm{E}_{\alpha,\ \alpha}\left(-\eta_{\alpha}(t-\varphi)^{\alpha}\right)\Upsilon(\varphi)d\varphi, \tag{5} $$

where $\eta_{\alpha} = \dfrac{\alpha}{1-\alpha}$ and

$$ \mathrm{AB}(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}. \tag{6} $$

For the MABC derivative, the Laplace transform is given by:

$$ \mathscr{L}\left\{{}^{\mathrm{MABC}}D_t^{\alpha}\Upsilon(t);\ p\right\} = \frac{\mathrm{AB}(\alpha)}{1-\alpha}\frac{p^{\alpha}\mathscr{L}\{Y(t);\ p\} - p^{\alpha-1}\Upsilon(0)}{p^{\alpha}+\eta_{\alpha}}, \left|\frac{\eta_{\alpha}}{p^{\alpha}}\right| < 1. \tag{7} $$

Definition 5 [12] For $t \geq 0$, the function Υ is defined. Then the Laplace transform of Υ, denoted by $\mathscr{L}\{Y\}$, is defined by the improper integral

$$ \mathscr{L}\{\Upsilon(t)\} = \mathrm{F}(p) = \int_0^{\infty} e^{-pt}\Upsilon(t)dt. \tag{8} $$

provided that the integral in (8) exists, i.e., that the integral is convergent.

The inverse Laplace transform is defined by

$$\mathscr{L}^{-\mathbf{1}}\{\Upsilon(t)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\, \mathrm{F}(p)dp. \tag{9}$$

Definition 6 [17] Let $\Upsilon \in H^1([0,T]), T > 0$, and $\alpha \in (0,1)$. For a function $\Upsilon(t)$ of order α, the ABC fractional derivative is given as

$${}^{\mathrm{ABC}}_{0}D_t^{\alpha}\Upsilon(t) = \frac{\mathrm{AB}(\alpha)}{1-\alpha}\int_0^t \frac{d}{dt}\Upsilon(\varphi)\mathrm{E}_{\alpha}\left(-\frac{\alpha(t-\varphi)^{\alpha}}{1-\alpha}\right)d\varphi, \tag{10}$$

The Mittag-Leffler kernel function of order α is denoted as $\mathrm{E}_{\alpha}(z)$ is which is defined as follows

$$\mathrm{E}_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^{\kappa}}{\Gamma(\alpha\kappa+1)}, \tag{11}$$

with $\mathrm{AB}(0) = \mathrm{AB}(1) = 1$ and $\mathrm{AB}(\alpha)$ denoting the normalization function. The Laplace transform can be derived as follows:

$$\mathscr{L}\left\{{}^{\mathrm{ABC}}D_t^{\alpha}\Upsilon(t)\right\} = \frac{\mathrm{AB}(\alpha)}{1-\alpha}\frac{p^{\alpha}\mathscr{L}\{\Upsilon(t)\} - p^{\alpha-1}\Upsilon(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}}, \quad \left|\frac{\alpha/(1-\alpha)}{p}\right| < 1.$$

3. Fractional order tumor growth model

Here, we present a memory-affected MABC model for tumor growth. The MABC scheme, which elucidates the above assumptions, is outlined as follows:

$$\begin{aligned}
{}^{\mathrm{MABC}}D_t^{\omega}\mathbf{u}(t) &= \mathbf{w}(t)q(\mathscr{D})\mathbf{u}(t) - h(\mathscr{D})\mathbf{u}(t),\\
{}^{\mathrm{MABC}}D_t^{\omega}\mathbf{v}(t) &= h(\mathscr{D})\mathbf{u}(t) - \mu_c\mathbf{v}(t),\\
{}^{\mathrm{MABC}}D_t^{\omega}\mathbf{w}(t) &= -\varpi\mathbf{w}(0)\mathbf{w}(t).
\end{aligned} \tag{12}$$

subject to initial conditions are

$$\mathbf{u}(0) = \mathbf{u}_0 = \mathscr{N}_1 \geq 0,\; \mathbf{v}(0) = \mathbf{v}_0 = \mathscr{N}_2 \geq 0,\; \mathbf{w}(0) = \mathbf{w}_0 = \mathscr{N}_3 \geq 0.$$

3.1 *Analysis for the suggested model*

In this study, we establish the existence of a nontrivial solution to the " homogenous fractional initial value problem". These formulas will be utilized:

$$\mathscr{L}\left(\mathrm{E}_{\omega}\left(ht^{\omega}\right)\right)=\frac{p^{\omega-1}}{p^{\omega}-h},\ \left|\frac{h}{p^{\omega}}\right|<1, \tag{13}$$

$$\mathscr{L}\left(t^{\omega-1}\mathrm{E}_{\omega,\ \omega}\left(ht^{\omega}\right)\right)=\frac{1}{p^{\omega}-h},\ \left|\frac{h}{p^{\omega}}\right|<1. \tag{14}$$

Lemma 1 [7–9] Let us analyze the "fractional initial value problem".

$$^{\mathrm{MABC}}D_{0}^{\omega}\mathbf{u}(t)=\zeta\mathbf{u},\ t>0,\ \mathbf{u}(0)=\mathbf{u}_{0},$$

$$^{\mathrm{MABC}}D_{0}^{\omega}\mathbf{v}(t)=\zeta\mathbf{v},\ t>0,\ \mathbf{v}(0)=\mathbf{v}_{0},$$

$$^{\mathrm{MABC}}D_{0}^{\omega}\mathbf{w}(t)=\zeta\mathbf{w},\ t>0,\ \mathbf{w}(0)=\mathbf{w}_{0}.$$

where $0<\omega<1$.

(1) For $\zeta=\frac{\mathrm{B}(\omega)}{1-\omega}$, the solution is given by

$$\mathbf{u}(t)=\mathbf{u}_{0}\begin{cases}-\dfrac{t^{-\omega}}{\vartheta_{\omega}\Gamma(1-\omega)},\ t\neq 0,\\ \\ 1,\ t=0.\end{cases}$$

$$\mathbf{v}(t)=\mathbf{v}_{0}\begin{cases}-\dfrac{t^{-\omega}}{\vartheta_{\omega}\Gamma(1-\omega)},\ t\neq 0,\\ \\ 1,\ t=0.\end{cases}$$

$$\mathbf{w}(t)=\mathbf{w}_{0}\begin{cases}-\dfrac{t^{-\omega}}{\vartheta_{\omega}\Gamma(1-\omega)},\ t\neq 0,\\ \\ 1,\ t=0.\end{cases}$$

(2) For $\zeta\neq\frac{B(\omega)}{1-\omega}$, the solution is given by

$$\mathbf{u}(t)=\mathbf{u}_{0}\begin{cases}\dfrac{\Xi_{\omega}\left(\vartheta_{\omega}\frac{\mathrm{f}_{\omega}}{1-f_{\omega}}t^{\omega}\right)}{1-\mathrm{f}_{\omega}},\ t\neq 0,\\ \\ 1,\ t=0.\end{cases}$$

$$\mathbf{v}(t)=\mathbf{v}_0\begin{cases}\dfrac{\Xi_\omega\left(\vartheta_\omega\frac{\mathrm{f}_\omega}{1-\mathrm{f}_\omega}t^\omega\right)}{1-f_\omega},\ t\neq 0,\\ \\ 1,\ t=0.\end{cases}$$

$$\mathbf{w}(t)=\mathbf{w}_0\begin{cases}\dfrac{\Xi_\omega\left(\vartheta_\omega\frac{\mathrm{f}_\omega}{1-\mathrm{f}_\omega}t^\omega\right)}{1-\mathrm{f}_\omega},\ t\neq 0,\\ \\ 1,\ t=0.\end{cases}$$

where $\mathbb{f}_\omega=\dfrac{\zeta(1-\omega)}{\mathrm{B}(\omega)}$.

Proof. (1) Given that

$$\int_0^t(t-k)^{\omega-1}\Xi_{\omega,\ \omega}\left(-\vartheta_\omega(t-k)^\omega\right)k^{-\omega}dk=\Gamma(1-\omega)\Xi_\omega\left(-\vartheta_\omega t^\omega\right), \tag{15}$$

for $t>0$, we obtain

$${}^{\mathrm{MABC}}D_0^\omega\mathbf{u}(t)=\frac{\mathrm{B}(\omega)}{1-\omega}\left(\mathbf{u}(t)-\Xi_\omega\left(-\vartheta_\omega t^\omega\right)\mathbf{u}_0-\vartheta_\omega\int_0^t(t-k)^{\omega-1}\Xi_{\omega,\omega}\left(-\vartheta_\omega(t-k)^\omega\right)\times\left(-\frac{\mathbf{u}_0}{\vartheta_\omega\Gamma(1-\omega)}k^{-\omega}\right)dk\right),$$

$${}^{\mathrm{MABC}}D_0^\omega\mathbf{v}(t)=\frac{\mathrm{B}(\omega)}{1-\omega}\left(\mathbf{v}(t)-\Xi_\omega\left(-\vartheta_\omega t^\omega\right)\mathbf{v}_0-\vartheta_\omega\int_0^t(t-k)^{\omega-1}\Xi_{\omega,\omega}\left(-\vartheta_\omega(t-k)^\omega\right)\times\left(-\frac{\mathbf{v}_0}{\vartheta_\omega\Gamma(1-\omega)}k^{-\omega}\right)dk\right),$$

$${}^{\mathrm{MABC}}D_0^\omega\mathbf{w}(t)=\frac{\mathrm{B}(\omega)}{1-\omega}\left(\mathbf{w}(t)-\Xi_\omega\left(-\vartheta_\omega t^\omega\right)\mathbf{w}_0-\vartheta_\omega\int_0^t(t-k)^{\omega-1}\Xi_{\omega,\omega}\left(-\vartheta_\omega(t-k)^\omega\right)\times\left(-\frac{\mathbf{w}_0}{\vartheta_\omega\Gamma(1-\omega)}k^{-\omega}\right)dk\right).$$

$$\begin{cases}=\dfrac{\mathrm{B}(\omega)}{1-\omega}\left(\mathbf{u}(t)-\Xi_\omega\left(-\vartheta_\omega t^\omega\right)\mathbf{u}_0+\Xi_\omega\left(-\vartheta_\omega t^\omega\right)\mathbf{u}_0\right),\\ \\ =\dfrac{\mathrm{B}(\omega)}{1-\omega}\left(\mathbf{v}(t)-\Xi_\omega\left(-\vartheta_\omega t^\omega\right)\mathbf{v}_0+\Xi_\omega\left(-\vartheta_\omega t^\omega\right)\mathbf{v}_0\right),\\ \\ =\dfrac{\mathrm{B}(\omega)}{1-\omega}\left(\mathbf{w}(t)-\Xi_\omega\left(-\vartheta_\omega t^\omega\right)\mathbf{w}_0+\Xi_\omega\left(-\vartheta_\omega t^\omega\right)\mathbf{w}_0\right).\end{cases}$$

$$\begin{cases} = \zeta \mathbf{u}(t), \\ = \zeta \mathbf{v}(t), \\ = \zeta \mathbf{w}(t). \end{cases}$$

This completes the proof.

(2) For $t > 0$, we can utilized equations (13) & (14)

$$\begin{cases} \mathscr{L}\left({}^{\mathrm{MABC}}D_0^{\omega}\mathbf{u}(t);\ p\right) = \dfrac{\mathrm{B}(\omega)}{1-\omega}\dfrac{1}{p^{\omega}+\vartheta_{\omega}}\left(\dfrac{\mathbf{u}_0 p^{\omega}}{1-\mathrm{f}_{\omega}} \times \dfrac{p^{\omega-1}}{p^{\omega}-\vartheta_{\omega}\frac{\mathrm{f}_{\omega}}{1-\mathrm{f}_{\omega}}} - \mathbf{u}_0 p^{\omega-1}\right), \\ \mathscr{L}\left({}^{\mathrm{MABC}}D_0^{\omega}\mathbf{v}(t);\ p\right) = \dfrac{\mathrm{B}(\omega)}{1-\omega}\dfrac{1}{p^{\omega}+\vartheta_{\omega}}\left(\dfrac{\mathbf{v}_0 p^{\omega}}{1-\mathrm{f}_{\omega}} \times \dfrac{p^{\omega-1}}{p^{\omega}-\vartheta_{\omega}\frac{\mathrm{f}_{\omega}}{1-\mathrm{f}_{\omega}}} - \mathbf{v}_0 p^{\omega-1}\right), \\ \mathscr{L}\left({}^{\mathrm{MABC}}D_0^{\omega}\mathbf{w}(t);\ p\right) = \dfrac{\mathrm{B}(\omega)}{1-\omega}\dfrac{1}{p^{\omega}+\vartheta_{\omega}}\left(\dfrac{\mathbf{w}_0 p^{\omega}}{1-\mathrm{f}_{\omega}} \times \dfrac{p^{\omega-1}}{p^{\omega}-\vartheta_{\omega}\frac{\mathrm{f}_{\omega}}{1-\mathrm{f}_{\omega}}} - \mathbf{w}_0 p^{\omega-1}\right). \end{cases}$$

$$\begin{cases} = \dfrac{\mathrm{B}(\omega)}{1-\omega}\mathbf{u}_0\dfrac{\mathrm{f}_{\omega}}{1-\mathrm{f}_{\omega}}\dfrac{p^{\omega-1}}{p^{\omega}-\vartheta_{\omega}\frac{\mathrm{f}_{\omega}}{1-\mathrm{f}_{\omega}}}, \\ = \dfrac{\mathrm{B}(\omega)}{1-\omega}\mathbf{V}_0\dfrac{\mathrm{f}_{\omega}}{1-\mathrm{f}_{\omega}}\dfrac{p^{\omega-1}}{p^{\omega}-\vartheta_{\omega}\frac{\mathrm{f}\omega}{1-\mathrm{f}_{\omega}}}, \\ = \dfrac{\mathrm{B}(\omega)}{1-\omega}\mathbf{W}_0\dfrac{\mathrm{f}_{\omega}}{1-\mathrm{f}_{\omega}}\dfrac{p^{\omega-1}}{p^{\omega}-\vartheta_{\omega}\frac{\mathrm{f}_{\omega}}{1-\mathrm{f}_{\omega}}}. \end{cases}$$

$$\begin{cases} = \zeta\dfrac{\mathbf{u}_0}{1-\mathrm{f}_{\omega}}\dfrac{p^{\omega-1}}{p^{\omega}-\vartheta_{\omega}\frac{\mathrm{f}_{\omega}}{1-\mathrm{f}_{\omega}}}, \\ = \zeta\dfrac{\mathbf{v}_0}{1-\mathrm{f}_{\omega}}\dfrac{p^{\omega-1}}{p^{\omega}-\vartheta_{\omega}\frac{\mathrm{f}_{\omega}}{1-\mathrm{f}_{\omega}}}, \\ = \zeta\dfrac{\mathbf{w}_0}{1-\mathrm{f}_{\omega}}\dfrac{p^{\omega-1}}{p^{\omega}-\vartheta_{\omega}\frac{\mathrm{f}_{\omega}}{1-\mathrm{f}_{\omega}}}. \end{cases}$$

$$\begin{cases} = \zeta \dfrac{\mathbf{u}_0}{1-\mathrm{f}_\omega} \mathscr{L}\left(\Xi_\omega\left(\vartheta_\omega \dfrac{\mathrm{f}_\omega}{1-\mathrm{f}_\omega} t^\omega\right)\right), \\ = \zeta \dfrac{\mathrm{v}_0}{1-\mathrm{f}_\omega} \mathscr{L}\left(\Xi_\omega\left(\vartheta_\omega \dfrac{\mathrm{f}_\omega}{1-\mathrm{f}_\omega} t^\omega\right)\right), \\ = \zeta \dfrac{\mathbf{w}_0}{1-\mathrm{f}_\omega} \mathscr{L}\left(\Xi_\omega\left(\vartheta_\omega \dfrac{\mathrm{f}_\omega}{1-\mathrm{f}_\omega} t^\omega\right)\right). \end{cases}$$

is complete the proof.

Lemma 2 [7–9] Examine the fractional differential equation

$$\begin{cases} {}^{\mathrm{MABC}}D_0^\omega \mathbf{u}(t) + \zeta \mathbf{u} = \mathscr{J}_1(t),\ t > 0,\ \mathbf{u}(0) = \mathbf{u}_0, \\ {}^{\mathrm{MABC}}D_0^\omega \mathbf{v}(t) + \zeta \mathbf{v} = \mathscr{J}_2(t),\ t > 0,\ \mathbf{v}(0) = \mathbf{v}_0, \\ {}^{\mathrm{MABC}}D_0^\omega \mathbf{w}(t) + \zeta \mathbf{w} = \mathscr{J}_3(t),\ t > 0,\ \mathbf{w}(0) = \mathbf{w}_0. \end{cases}$$

For $0 < \omega < 1$, and $\zeta \neq -\dfrac{\mathrm{B}(\omega)}{1-\omega}$, the solutions to the previous “fractional initial value problem” is as follows:

$$\mathbf{u}(t) = \begin{cases} \widehat{\mathbf{u}},\ \mathrm{t} \neq 0, \\ \mathbf{u}_0,\ t = 0, \end{cases}$$

$$\mathbf{v}(t) = \begin{cases} \hat{\mathbf{v}},\ \mathrm{t} \neq 0, \\ \mathbf{v}_0,\ t = 0, \end{cases} \tag{16}$$

$$\mathbf{w}(t) = \begin{cases} \widehat{\mathbf{w}},\ \mathrm{t} \neq 0, \\ \mathbf{w}_0,\ t = 0. \end{cases}$$

where

$$\begin{cases} \widehat{\mathbf{u}} = \mathbf{u}_0 \dfrac{\mathrm{B}(\omega)}{l_\omega} \Xi_\omega \left(-\dfrac{\zeta\omega}{l_\omega} t^\omega\right) + \dfrac{1-\omega}{l_\omega} \mathscr{J}_1(t) + \dfrac{1-\omega}{l_\omega} \left(\vartheta_\omega - \dfrac{\zeta\omega}{l_\omega}\right) \left(t^{\omega-1} \Xi_{\omega,\ \omega} \left(-\dfrac{\zeta\omega}{l_\omega} t^\omega\right)\right) \mathscr{J}_1, \\ \hat{\mathbf{v}} = \mathbf{v}_0 \dfrac{\mathrm{B}(\omega)}{l_\omega} \Xi_\omega \left(-\dfrac{\zeta\omega}{l_\omega} t^\omega\right) + \dfrac{1-\omega}{l_\omega} \mathscr{J}_2(t) + \dfrac{1-\omega}{l_\omega} \left(\vartheta_\omega - \dfrac{\zeta\omega}{l_\omega}\right) \left(t^{\omega-1} \Xi_{\omega,\ \omega} \left(-\dfrac{\zeta\omega}{l_\omega} t^\omega\right)\right) \mathscr{J}_2, \\ \widehat{\mathbf{w}} = \mathbf{w}_0 \dfrac{\mathrm{B}(\omega)}{l_\omega} \Xi_\omega \left(-\dfrac{\zeta\omega}{l_\omega} t^\omega\right) + \dfrac{1-\omega}{l_\omega} \mathscr{J}_3(t) + \dfrac{1-\omega}{l_\omega} \left(\vartheta_\omega - \dfrac{\zeta\omega}{l_\omega}\right) \left(t^{\omega-1} \Xi_{\omega,\ \omega} \left(-\dfrac{\zeta\omega}{l_\omega} t^\omega\right)\right) \mathscr{J}_3. \end{cases}$$

and $l_\omega = \mathrm{B}(\omega) + \zeta(1-\omega)$.

Proof. By employing equations (13) (14), it is straightforward to prove that

$$\begin{cases} \mathscr{L}(\widehat{\mathbf{u}}; p) = \dfrac{\mathbf{u}_0\ \mathrm{B}(\omega) p^{\omega-1} + (1-\omega)(p^\omega + \vartheta_\omega) \mathscr{L}(\mathscr{J}_1; p)}{l_\omega p^\omega + \zeta\omega}, \\ \mathscr{L}(\hat{\mathbf{v}}; p) = \dfrac{\mathbf{v}_0\ \mathrm{B}(\omega) p^{\omega-1} + (1-\omega)(p^\omega + \vartheta_\omega) \mathscr{L}(\mathscr{J}_2; p)}{l_\omega p^\omega + \zeta\omega}, \\ \mathscr{L}(\widehat{\mathbf{w}}; p) = \dfrac{\mathbf{w}_0\ \mathrm{B}(\omega) p^{\omega-1} + (1-\omega)(p^\omega + \vartheta_\omega) \mathscr{L}(\mathscr{J}_3; p)}{l_\omega p^\omega + \zeta\omega}. \end{cases} \tag{17}$$

By Eq. (9) we have

$$\begin{cases} \mathscr{L}\left({}^{\mathrm{MABC}}D_0^\omega \mathbf{u}(t) + \zeta\mathbf{u};\ p\right) = \dfrac{\mathrm{B}(\omega)}{1-\omega} \dfrac{p^\omega \mathscr{L}(\widehat{\mathbf{u}};\ p) - p^{\omega-1}\mathbf{u}_0}{p^\omega + \vartheta_\omega} + \zeta\mathscr{L}(\widehat{\mathbf{u}};\ p), \\ \mathscr{L}\left({}^{\mathrm{MABC}}D_0^\omega \mathbf{v}(t) + \zeta\mathbf{v};\ p\right) = \dfrac{\mathrm{B}(\omega)}{1-\omega} \dfrac{p^\omega \mathscr{L}(\hat{\mathbf{v}};\ p) - p^{\omega-1}\mathbf{v}_0}{p^\omega + \vartheta_\omega} + \zeta\mathscr{L}(\hat{\mathbf{v}};\ p), \\ \mathscr{L}\left({}^{\mathrm{MABC}}D_0^\omega \mathbf{w}(t) + \zeta\mathbf{W};\ p\right) = \dfrac{\mathrm{B}(\omega)}{1-\omega} \dfrac{p^\omega \mathscr{L}(\widehat{\mathbf{w}};\ p) - p^{\omega-1}\mathbf{w}_0}{p^\omega + \vartheta_\omega} + \zeta\mathscr{L}(\widehat{\mathbf{w}};\ p). \end{cases} \tag{18}$$

Direct computation will result in

$$
\begin{cases}
\mathscr{L}\left({}^{\text{MABC}}D_0^{\omega}\mathbf{u}(t)+\zeta\mathbf{u};\ p\right)=\dfrac{1}{(1-\omega)(p^{\omega}+\vartheta_{\omega})}\left((l_{\omega}p^{\omega}+\zeta\omega)\mathscr{L}(\widehat{\mathbf{u}};\ p)-\mathrm{B}(\omega)p^{\omega-1}\mathbf{u}_0\right),\\[2ex]
\mathscr{L}\left({}^{\text{MABC}}D_0^{\omega}\mathbf{v}(t)+\zeta\mathbf{v};\ p\right)=\dfrac{1}{(1-\omega)(p^{\omega}+\vartheta_{\omega})}\left((l_{\omega}p^{\omega}+\zeta\omega)\mathscr{L}(\hat{\mathbf{v}};\ p)-\mathrm{B}(\omega)p^{\omega-1}\mathbf{v}_0\right),\\[2ex]
\mathscr{L}\left({}^{\text{MABC}}D_0^{\omega}\mathbf{w}(t)+\zeta\mathbf{w};\ p\right)=\dfrac{1}{(1-\omega)(p^{\omega}+\vartheta_{\omega})}\left((l_{\omega}p^{\omega}+\zeta\omega)\mathscr{L}(\widehat{\mathbf{w}};\ p)-\mathrm{B}(\omega)p^{\omega-1}\mathbf{w}_0\right).
\end{cases}
\tag{19}
$$

equation (17) is inserted into equation (19) to provide the subsequent outcome:

$$
\begin{cases}
\mathscr{L}\left({}^{\text{MABC}}D_0^{\omega}\mathbf{u}(t)+\zeta\mathbf{u};\ p\right)=\dfrac{1}{(1-\omega)(p^{\omega}+\vartheta_{\omega})}\left(\mathrm{B}(\omega)p^{\omega-1}\mathbf{u}_0+(1-\omega)(p^{\omega}+\vartheta_{\omega})\mathscr{L}(\mathscr{J}_1;\ p)-\mathrm{B}(\omega)p^{\omega-1}\mathbf{u}_0\right),\\[2ex]
\mathscr{L}\left({}^{\text{MABC}}D_0^{\omega}\mathbf{v}(t)+\zeta\mathbf{v};\ p\right)=\dfrac{1}{(1-\omega)(p^{\omega}+\vartheta_{\omega})}\left(\mathrm{B}(\omega)p^{\omega-1}\mathbf{v}_0+(1-\omega)(p^{\omega}+\vartheta_{\omega})\mathscr{L}(\mathscr{J}_2;\ p)-\mathrm{B}(\omega)p^{\omega-1}\mathbf{v}_0\right),\\[2ex]
\mathscr{L}\left({}^{\text{MABC}}D_0^{\omega}\mathbf{w}(t)+\zeta\mathbf{w};\ p\right)=\dfrac{1}{(1-\omega)(p^{\omega}+\vartheta_{\omega})}\left(\mathrm{B}(\omega)p^{\omega-1}\mathbf{w}_0+(1-\omega)(p^{\omega}+\vartheta_{\omega})\mathscr{L}(\mathscr{J}_3;\ p)-\mathrm{B}(\omega)p^{\omega-1}\mathbf{w}_0\right).
\end{cases}
$$

$$
\begin{cases}
=\mathscr{L}(\mathscr{J}_1;\ p),\\[2ex]
=\mathscr{L}(\mathscr{J}_2;\ p),\\[2ex]
=\mathscr{L}(\mathscr{J}_3;\ p).
\end{cases}
$$

The proof has been completed.

Remark If $\mathscr{J}_i\in\mathscr{C}[0,\ \mathrm{T}]$ in which $i=1,\ 2,\ 3$, then

$$
\begin{cases}
\widehat{\mathbf{u}}=\dfrac{1}{l_{\omega}}\left(\mathbf{u}_0\,\mathrm{B}(\omega)+(1-\omega)\mathscr{J}_1(0)\right),\\[2ex]
\hat{\mathbf{v}}=\dfrac{1}{l_{\omega}}\left(\mathbf{v}_0\,\mathrm{B}(\omega)+(1-\omega)\mathscr{J}_2(0)\right),\\[2ex]
\widehat{\mathbf{w}}=\dfrac{1}{l_{\omega}}\left(\mathbf{w}_0\,\mathrm{B}(\omega)+(1-\omega)\mathscr{J}_3(0)\right).
\end{cases}
$$

Adding more conditions

$$\begin{cases} \zeta \mathbf{u}_0 = \mathscr{J}_1(0), \\ \zeta \mathbf{v}_0 = \mathscr{J}_2(0), \\ \zeta \mathbf{w}_0 = \mathscr{J}_3(0). \end{cases}$$

then $\widehat{\mathbf{u}} = \mathbf{u}_0$, $\hat{\mathbf{v}} = \mathbf{v}_0$ and $\widehat{\mathbf{w}} = \mathbf{w}_0$, eq. (16) provides a solution that is continuous. The previously mentioned condition is necessary for ensuring the existence of a solution.

4. Iteration scheme and stability analysis

When the Laplace transform is applied to each side of model (12), the outcome of the transformation is as follows:

$$\begin{aligned} &\mathscr{L}\left({}^{\text{MABC}}D_0^{\omega}\mathbf{u}(t)\right) = \mathscr{L}\{\mathbf{w}(t)q(\mathscr{D})\mathbf{u}(t) - h(\mathscr{D})\mathbf{u}(t)\}, \\ &\mathscr{L}\left({}^{\text{MABC}}D_0^{\omega}\mathbf{v}(t)\right) = \mathscr{L}\left\{h(\mathscr{D})\mathbf{u}(t) - \mu_c\mathbf{v}(t)\right\}, \\ &\mathscr{L}\left({}^{\text{MABC}}D_0^{\omega}\mathbf{w}(t)\right) = \mathscr{L}\{-\varpi\mathbf{w}(0)\mathbf{w}(t)\}. \end{aligned} \tag{20}$$

or

$$\begin{aligned} &\frac{\text{B}(\omega)}{1-\omega} \times \frac{p^{\omega}\mathscr{L}\{\mathbf{u}(t)\} - \mathbf{u}(0)p^{\omega-1}}{p^{\omega}+\gamma_{\omega}} = \mathscr{L}\{\mathbf{w}(t)q(\mathscr{D})\mathbf{u}(t) - h(\mathscr{D})\mathbf{u}(t)\}, \\ &\frac{\text{B}(\omega)}{1-\omega} \times \frac{p^{\omega}\mathscr{L}\{\mathbf{v}(t)\} - \mathbf{v}(0)p^{\omega-1}}{p^{\omega}+\gamma_{\omega}} = \mathscr{L}\left\{h(\mathscr{D})\mathbf{u}(t) - \mu_c\mathbf{v}(t)\right\}, \\ &\frac{\text{B}(\omega)}{1-\omega} \times \frac{p^{\omega}\mathscr{L}\{\mathbf{w}(t)\} - \mathbf{w}(0)p^{\omega-1}}{p^{\omega}+\gamma_{\omega}} = \mathscr{L}\{-\varpi\mathbf{w}(0)\mathbf{w}(t)\}. \end{aligned} \tag{21}$$

with primary condition, we obtain

$$\begin{aligned} &\mathscr{L}(\mathbf{u}(t)) = \frac{\mathbf{u}_0}{p} + \left[\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\text{B}(\omega)p^{\omega}}\mathscr{L}\{\mathbf{w}(t)q(\mathscr{D})\mathbf{u}(t) - h(\mathscr{D})\mathbf{u}(t)\}\right], \\ &\mathscr{L}(\mathbf{v}(t)) = \frac{\mathbf{v}_0}{p} + \left[\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\text{B}(\omega)p^{\omega}}\mathscr{L}\left\{h(\mathscr{D})\mathbf{u}(t) - \mu_c\mathbf{v}(t)\right\}\right], \\ &\mathscr{L}(\mathbf{w}(t)) = \frac{\mathbf{w}_0}{p} + \left[\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\text{B}(\omega)p^{\omega}}\mathscr{L}\{-\varpi\mathbf{w}(0)\mathbf{w}(t)\}\right]. \end{aligned} \tag{22}$$

Let us consider the solution $\mathbf{u}(\mathrm{t}), \mathbf{v}(\mathrm{t})$, and $\mathbf{w}(\mathrm{t})$ which are expressed as an infinite series.

$$\mathbf{u}(t)=\sum_{\mathrm{q}=0}^{\infty}\mathbf{u}_{\mathrm{q}},\ \mathbf{v}(t)=\sum_{\mathrm{q}=0}^{\infty}\mathbf{v}_{\mathrm{q}},\ \mathbf{w}(t)=\sum_{\mathrm{q}=0}^{\infty}\mathbf{w}_{\mathrm{q}}. \tag{23}$$

we resolve nonlinear term as follows:

$$\mathbf{u}(t)\mathbf{v}(t)=\sum_{q=0}^{\infty}\mathrm{G}_{\mathrm{q}},\mathbf{u}(t)\mathbf{w}(t)=\sum_{q=0}^{\infty}\mathrm{H}_{\mathrm{q}}. \tag{24}$$

where G_{q} and H_{q} additionally decompose into the following parts:

$$\begin{aligned}
\mathrm{G}_{\mathrm{q}} &= \frac{1}{\Gamma(\mathrm{q}+1)}\frac{d^{\mathrm{q}}}{d\delta^{\mathrm{q}}}\left[\sum_{\mathrm{j}=0}^{\mathrm{q}}\delta^{\mathrm{j}}\mathbf{u}_{\mathrm{j}}(t)\sum_{\mathrm{j}=0}^{\mathrm{q}}\delta^{\mathrm{j}}\mathbf{v}_{\mathrm{j}}(t)\right]\Bigg|_{\delta=0}, \\
\mathrm{H}_{\mathrm{q}} &= \frac{1}{\Gamma(\mathrm{q}+1)}\frac{d^{\mathrm{q}}}{d\delta^{\mathrm{q}}}\left[\sum_{\mathrm{j}=0}^{\mathrm{q}}\delta^{\mathrm{j}}\mathbf{u}_{\mathrm{j}}(t)\sum_{\mathrm{j}=0}^{\mathrm{q}}\delta^{\mathrm{j}}\mathbf{w}_{\mathrm{j}}(t)\right]\Bigg|_{\delta=0}.
\end{aligned} \tag{25}$$

When Eq. (23) and Eq. (24) are substituted into (22), the result is

$$\begin{aligned}
&\mathscr{L}\left\{\sum_{\mathrm{q}=0}^{\infty}\mathbf{u}_{\mathrm{q}}\right\}=\frac{\mathbf{u}_0}{p}+\left[\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}\left\{q(\mathscr{D})\sum_{\mathrm{q}=0}^{\infty}\mathrm{H}_{\mathrm{q}}-h(\mathscr{D})\sum_{\mathrm{q}=0}^{\infty}\mathbf{u}_{\mathrm{q}}\right\}\right], \\
&\mathscr{L}\left(\sum_{\mathrm{q}=0}^{\infty}\mathbf{v}_{\mathrm{q}}\right)=\frac{\mathbf{v}_0}{p}+\left[\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}\left\{h(\mathscr{D})\sum_{\mathrm{q}=0}^{\infty}\mathbf{u}_{\mathrm{q}}-\mu_{\mathrm{c}}\sum_{\mathrm{q}=0}^{\infty}\mathbf{v}_{\mathrm{q}}\right\}\right], \\
&\mathscr{L}\left(\sum_{\mathrm{q}=0}^{\infty}\mathbf{w}_{\mathrm{q}}\right)=\frac{\mathbf{w}_0}{p}+\left[\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}\left\{-\varpi\mathbf{w}(0)\sum_{\mathrm{q}=0}^{\infty}\mathbf{w}_{\mathrm{q}}\right\}\right].
\end{aligned} \tag{26}$$

The Iteration process generated through the matching of both sides of Eq. (26):

$$\mathscr{L}\{\mathbf{u}_0\}=\frac{\mathscr{N}_1}{p},$$

$$\mathscr{L}\{\mathbf{u}_1\}=\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}\{q(\mathscr{D})\mathrm{H}_0-h(\mathscr{D})\mathbf{u}_0\},$$

$$\mathscr{L}\{\mathbf{u}_2\}=\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}\{q(\mathscr{D})\mathrm{H}_1-h(\mathscr{D})\mathbf{u}_1\},$$

.

.

$$\mathscr{L}\left\{\mathbf{u}_{\mathrm{q}+1}\right\}=\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\mathrm{B}(\omega) p^{\omega}} \mathscr{L}\left\{q(\mathscr{D}) \mathrm{H}_{\mathrm{q}}-h(\mathscr{D}) \mathbf{u}_{\mathrm{q}}\right\}, \ \mathrm{q} \geq 1, \tag{27}$$

$$\mathscr{L}\left\{\mathbf{v}_{0}\right\}=\frac{\mathscr{N}_{2}}{p},$$

$$\mathscr{L}\left\{\mathbf{v}_{1}\right\}=\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\mathrm{B}(\omega) p^{\omega}} \mathscr{L}\left\{h(\mathscr{D}) \mathbf{u}_{0}-\mu_{\mathrm{c}} \mathbf{v}_{0}\right\},$$

$$\mathscr{L}\left\{\mathbf{v}_{2}\right\}=\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\mathrm{B}(\omega) p^{\omega}} \mathscr{L}\left(\left\{h(\mathscr{D}) \mathbf{u}_{1}-\mu_{\mathrm{c}} \mathbf{v}_{1}\right\}\right),$$

.

.

$$\mathscr{L}\left\{\mathbf{v}_{\mathrm{q}+1}\right\}=\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\mathrm{B}(\omega) p^{\omega}} \mathscr{L}\left\{h(\mathscr{D}) \mathbf{u}_{\mathrm{q}}-\mu_{\mathrm{c}} \mathbf{v}_{\mathrm{q}}\right\}, \ q \geq 1, \tag{28}$$

$$\mathscr{L}\left\{\mathbf{w}_{0}\right\}=\frac{\mathscr{N}_{3}}{p},$$

$$\mathscr{L}\left\{\mathbf{w}_{1}\right\}=\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\mathrm{B}(\omega) p^{\omega}} \mathscr{L}\left\{-\varpi \mathbf{w}(0) \mathbf{w}_{0}\right\},$$

$$\mathscr{L}\left\{\mathbf{w}_{2}\right\}=\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{\mathrm{B}(\omega) p^{\omega}} \mathscr{L}\left\{-\varpi \mathbf{w}(0) \mathbf{w}_{1}\right\},$$

.

.

$$\mathscr{L}\left\{\mathbf{w}_{q+1}\right\}=\frac{(1-\omega)\left(p^{\omega}+\gamma_{\omega}\right)}{B(\omega) p^{\omega}} \mathscr{L}\left\{-\varpi \mathbf{w}(0) \mathbf{w}_{\mathrm{q}}\right\}, \ \mathrm{q} \geq 1. \tag{29}$$

Upon evaluating the initial three terms and performing the inverse Laplace transform on equations (27-29), the resulting expression is obtained.

$$
\begin{aligned}
\mathbf{u}_0 =& \mathscr{N}_1, \\
\mathbf{u}_1 =& \left(1-\omega+\frac{t^{\omega}}{\Gamma(\omega)}\right)\frac{1}{\mathrm{B}(\omega)}\left(q(\mathscr{D})\mathscr{N}_1\mathscr{N}_3-h(\mathscr{D})\mathscr{N}_1\right), \\
\mathbf{u}_2 =& \left[\left(1-\omega+\frac{t^{\omega}}{\Gamma(\omega)}\right)\frac{1}{\mathrm{B}(\omega)}\right]^2\left(\left(q(\mathscr{D})\mathscr{N}_1\mathscr{N}_3-h(\mathscr{D})\mathscr{N}_1\right)\times\left(\mathscr{D}(\mathscr{D})\mathscr{N}_3-h(\mathscr{D})\right)\right), \\
&+q(\mathscr{D})\mathscr{N}_1\left[\left(1-\omega+\frac{t^{\omega}}{\Gamma(\omega)}\right)\frac{1}{\mathrm{B}(\omega)}\right]^2\left(\mathscr{N}_3-\varpi\mathbf{w}(0)\mathscr{N}_3\right), \\
\mathbf{v}_0 =& \mathscr{N}_2, \\
\mathbf{v}_1 =& \left(1-\omega+\frac{t^{\omega}}{\Gamma(\omega)}\right)\frac{1}{\mathrm{B}(\omega)}\left(h(\mathscr{D})\mathscr{N}_1-\mu_{\mathrm{c}}\mathscr{N}_2\right), \\
\mathbf{v}_2 =& \left[\left(1-\omega+\frac{t^{\omega}}{\Gamma(\omega)}\right)\frac{1}{\mathrm{B}(\omega)}\right]^2\left(h(\mathscr{D})\left(\left(q(\mathscr{D})\mathscr{N}_1\mathscr{N}_3-h(\mathscr{D})\mathscr{N}_1\right)\right)-\mu_{\mathrm{c}}\left(h(\mathscr{D})\mathscr{N}_1-\mu_{\mathrm{c}}\mathscr{N}_2\right)\right), \\
\mathbf{w}_0 =& \mathscr{N}_3, \\
\mathbf{w}_1 =& \left(1-\omega+\frac{t^{\omega}}{\Gamma(\omega)}\right)\frac{1}{\mathrm{B}(\omega)}\left(-\varpi\mathbf{w}(0)\mathscr{N}_3\right), \\
\mathbf{w}_2 =& \left[\left(1-\omega+\frac{t^{\omega}}{\Gamma(\omega)}\right)\frac{1}{\mathrm{B}(\omega)}\right]^2\left(-\varpi\mathbf{w}(0)\left(-\varpi\mathbf{w}(0)\mathscr{N}_3\right)\right).
\end{aligned} \tag{30}
$$

and so on.

The remaining terms can be computed using this method. Ultimately, the solutions that are required can be presented in the following manner:

$$
\begin{aligned}
\mathbf{u}(t) =& \mathscr{N}_1+\left(1-\omega+\frac{t^{\omega}}{\Gamma(\omega)}\right)\frac{1}{\mathrm{B}(\omega)}\left(q(\mathscr{D})\mathscr{N}_1\mathscr{N}_3-h(\mathscr{D})\mathscr{N}_1\right)+\left[\left(1-\omega+\frac{t^{\omega}}{\Gamma(\omega)}\right)\frac{1}{\mathrm{B}(\omega)}\right]^2\left((q(\mathscr{D})\mathscr{N}_1\mathscr{N}_3-\right. \\
&\left. h(\mathscr{D})\mathscr{N}_1)\times\left(q(\mathscr{D})\mathscr{N}_3-h(\mathscr{D})\right)\right)+q(\mathscr{D})\mathscr{N}_1\left[\left(1-\omega+\frac{t^{\omega}}{\Gamma(\omega)}\right)\frac{1}{\mathrm{B}(\omega)}\right]^2\left(\mathscr{N}_3-\varpi\mathbf{w}(0)\mathscr{N}_3\right),
\end{aligned} \tag{31}
$$

$$\mathbf{v}(t) = \mathscr{N}_2 + \left(1 - \omega + \frac{t^{\omega}}{\Gamma(\omega)}\right) \frac{1}{\mathrm{B}(\omega)} \left(h(\mathscr{D})\mathscr{N}_1 - \mu_{\mathrm{c}}\mathscr{N}_2\right) \tag{32}$$

$$+ \left[\left(1 - \omega + \frac{t^{\omega}}{\Gamma(\omega)}\right) \frac{1}{\mathrm{B}(\omega)}\right]^2 \left(h(\mathscr{D})\left((q(\mathscr{D})\mathscr{N}_1\mathscr{N}_3 - h(\mathscr{D})\mathscr{N}_1)\right) - \mu_{\mathrm{c}}\left(h(\mathscr{D})\mathscr{N}_1 - \mu_{\mathrm{c}}\mathscr{N}_2\right)\right),$$

$$\mathbf{w}(t) = \mathscr{N}_3 + \left(1 - \omega + \frac{t^{\omega}}{\Gamma(\omega)}\right) \frac{1}{\mathrm{B}(\omega)} \left(-\omega \mathbf{w}(0)\mathscr{N}_3\right) + \left[\left(1 - \omega + \frac{t^{\omega}}{\Gamma(\omega)}\right) \frac{1}{\mathrm{B}(\omega)}\right]^2 \left(-\omega \mathbf{w}(0)\left(-\omega \mathbf{w}(0)\mathscr{N}_3\right)\right). \tag{33}$$

Theorem 4.1 Let $\mathbb{B}$ be a Banach space with a norm $|.|$, and let $\mathbb{K}: \ \mathbb{B} \to \mathbb{B}$ be a map that fulfills the following conditions:

$$\|\mathbb{K}_x - \mathbb{K}_y\| \leq \Lambda \|X - \mathbb{K}_x\| + \pi\|x - y\|,$$

for all $x, \ y \in \mathbb{B}$, where $0 \leq \Lambda, \ 0 \leq \pi < 1$. Thus, $\mathbb{K}$ becomes Picard $\mathbb{K}$-stable.

Theorem 4.2 Define $\mathbb{K}$ as a self-map as follows:

$$\mathbb{K}[\mathbf{u}_{\mathrm{q}}(t)] = \mathbf{u}_{\mathrm{q}+1}(t) = \mathbf{u}_n(0) + \mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega} + \gamma_{\omega})}{\mathrm{B}(\omega)p^{\omega}} \mathscr{L}\left\{\mathbf{w}_{\mathrm{q}}\mathscr{Z}(\mathscr{D})\mathbf{u}_{\mathrm{q}} - h(\mathscr{D})\mathbf{u}_{\mathrm{q}}\right\}\right],$$

$$\mathbb{K}[\mathbf{v}_{\mathrm{q}}(t)] = \mathbf{v}_{\mathrm{q}+1}(t) = \mathbf{v}_n(0) + \mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega} + \gamma_{\omega})}{\mathrm{B}(\omega)p^{\omega}} \mathscr{L}\left\{(\langle\rangle)(\mathscr{D})\mathbf{u}_{\mathrm{q}} - \mu_{\mathrm{c}}\mathbf{v}_{\mathrm{q}}\right\}\right], \tag{34}$$

$$\mathbb{K}[\mathbf{w}_{\mathrm{q}}(t)] = \mathbf{w}_{\mathrm{q}+1}(t) = \mathbf{w}_n(0) + \mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega} + \gamma_{\omega})}{\mathrm{B}(\omega)p^{\omega}} \mathscr{L}\left\{-\varpi \mathbf{w}(0)\mathbf{w}_{\mathrm{q}}\right\}\right].$$

Consequently, the iteration attains $\mathbb{K}$-stability in $\mathscr{L}^1(x, \ y)$ when the subsequent conditions are full filled:

$$(1 + q(\mathscr{D})(\mathrm{N}_1 + \mathrm{N}_3)\phi_1(v) - h(\mathscr{D})\phi_2(v)) < 1,$$

$$(1 + h(\mathscr{D})\phi_2(v) - \mu_c\phi_3(v)) < 1, \tag{35}$$

$$(1 - \varpi \mathbf{w}(0)\phi_4(v)) < 1.$$

Proof. In order to demonstrate the existence of a fixed point for T, we computed the subsequent expression for (q, p) $\in \mathbb{N} \times \mathbb{N}$:

$$\mathbb{K}[\mathbf{u}_{\mathrm{q}}(t)]-\mathbb{K}[\mathbf{u}_{\mathrm{p}}(t)] = \mathbf{u}_{\mathrm{q}}(t)-\mathbf{u}_{\mathrm{p}}(t)+\mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega}+\gamma_{\omega})}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}\{\mathbf{w}_{\mathrm{q}}q(\mathscr{D})\mathbf{u}_{\mathrm{q}}-h(\mathscr{D})\mathbf{u}_{\mathrm{q}}\}\right]$$

$$-\mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega}+\gamma_{\omega})}{B(\omega)p^{\omega}}\mathscr{L}\{\mathbf{w}_{p}q(\mathscr{D})\mathbf{u}_{p}-h(\mathscr{D})\mathbf{u}_{p}\}\right],$$

$$\mathbb{K}[\mathbf{v}_{\mathbf{q}}(t)]-\mathbb{K}[\mathbf{v}_{\mathbf{p}}(t)] = \mathbf{v}_{\mathrm{q}}(t)-\mathbf{v}_{\mathrm{p}}(t)+\mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega}+\gamma_{\omega})}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}\{\langle(\mathscr{D})\mathbf{u}_{\mathrm{q}}-\mu_{\mathrm{c}}\mathbf{v}_{\mathrm{q}}\}\right]$$

$$-\mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega}+\gamma_{\omega})}{B(\omega)p^{\omega}}\mathscr{L}\{h(\mathscr{D})\mathbf{u}_{p}-\mu_{\mathrm{c}}\mathbf{v}_{\mathrm{p}}\}\right], \tag{36}$$

$$\mathbb{K}[\mathbf{w}_{\mathrm{q}}(t)]-\mathbb{K}[\mathbf{w}_{\mathrm{p}}(t)] = \mathbf{w}_{\mathrm{q}}(t)-\mathbf{w}_{\mathrm{p}}(t)+\mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega}+\gamma_{\omega})}{B(\omega)p^{\omega}}\mathscr{L}\{-\varpi\mathbf{w}(0)\mathbf{w}_{\mathrm{q}}\}\right]$$

$$-\mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega}+\gamma_{\omega})}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}\{-\varpi\mathbf{w}(0)\mathbf{w}_{\mathrm{v}}\}\right].$$

By computing the norm of each side of the 1^{st} Eq. of (36) we derive

$$\|\mathbb{K}(\mathbf{u}_{\mathrm{q}}(t))-\mathbb{K}(\mathbf{u}_{p}(t))\| = \left\| \begin{array}{l} \mathbf{u}_{\mathrm{q}}(t)-\mathbf{u}_{p}(t)+\mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega}+\gamma_{\omega})}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}\{\mathbf{w}_{\mathrm{q}}q(\mathscr{D})\mathbf{u}_{\mathrm{q}}-h(\mathscr{D})\mathbf{u}_{\mathrm{q}}\}\right] \\ -\mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega}+\gamma_{\omega})}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}\{\mathbf{w}_{p}q(\mathscr{D})\mathbf{u}_{p}-h(\mathscr{D})\mathbf{u}_{p}\}\right] \end{array} \right\|. \tag{37}$$

By applying the triangular inequality and simplifying Eq. (37), we obtain

$$\|\mathbb{K}(\mathbf{u}_{\mathrm{q}}(t))-\mathbb{K}(\mathbf{u}_{p}(t))\| \le \|\mathbf{u}_{\mathrm{q}}(t)-\mathbf{u}_{p}(t)\|+\mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega}+\gamma_{\omega})}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}[\|q(\mathscr{D})\mathbf{u}_{p}(\mathbf{w}_{\mathrm{q}}-\mathbf{w}_{\mathrm{p}})\|\right.$$

$$\left.+\|q(\mathscr{D})\mathbf{w}_{\mathrm{q}}(\mathbf{u}_{\mathrm{q}}-\ \mathbf{u}_{p})\|+\|-h(\mathscr{D})(\mathbf{u}_{\mathrm{q}}-\mathbf{u}_{p})\|]\right]. \tag{38}$$

Considering the respective impact of both solutions, we obtain

$$\|\mathbf{u}_{\mathrm{q}}(t)-\mathbf{u}_{\mathrm{p}}(t)\| \cong \|\mathbf{v}_{\mathrm{q}}(t)-\mathbf{v}_{\mathrm{p}}(t)\| \cong \|\mathbf{w}_{\mathrm{q}}(t)-\mathbf{w}_{\mathrm{p}}(t)\|. \tag{39}$$

By substituting this value into Eq. (38), we derive the following relationship:

$$
\begin{aligned}
\|\mathbb{K}(\mathbf{u}_{\mathfrak{q}}(t))-\mathbb{K}(\mathbf{u}_p(t))\| \leq \|\mathbf{u}_{\mathfrak{q}}(t)-\mathbf{u}_p(t)\|+\mathscr{L}^{-1}\left[\frac{(1-\omega)(p^{\omega}+\gamma_{\omega})}{B(\omega)p^{\omega}}\mathscr{L}\left[\|q(\mathscr{D})\mathbf{u}_p(\mathbf{u}_{\mathfrak{q}}-\mathbf{u}_p)\|\right.\right.\\
\left.\left.+\|q(\mathscr{D})\mathbf{w}_{\mathfrak{q}}(\mathbf{u}_{\mathfrak{q}}-\ \mathbf{u}_p)\|+\|-h(\mathscr{D})(\mathbf{u}_{\mathfrak{q}}-\mathbf{u}_p)\|\right]\right].
\end{aligned} \tag{40}
$$

The convergent sequence $\mathbf{u}_{\mathfrak{q}}$, $\mathbf{v}_{\mathfrak{q}}$ and $\mathbf{w}_{\mathfrak{q}}$ are bounded. Subsequently we can acquire four different positive constant, namely $\mathscr{N}_1$, $\mathscr{N}_2$ and $\mathscr{N}_3$ that hold true $\forall$ values of t.

$$
\|\mathbf{u}_{\mathfrak{p}}\|<\mathscr{N}_1,\ \|\mathbf{v}_{\mathfrak{p}}\|<\mathscr{N}_2,\ \|\mathbf{w}_{\mathfrak{p}}\|<\mathscr{N}_3,\ (\mathfrak{q},\ \mathfrak{p})\in\mathbb{N}\times\mathbb{N}. \tag{41}
$$

Moreover, when equations (40) and (41) are involved, we obtain

$$
\|\mathbb{K}(\mathbf{u}_{\mathfrak{q}}(t))-\mathbb{K}(\mathbf{u}_p(t))\| \leq (1+q(\mathscr{D})(\mathrm{N}_1+\mathrm{N}_3)\phi_1(v)-h(\mathscr{D})\phi_2(v))\|(\mathbf{u}_{\mathfrak{q}}-\mathbf{u}_p)\|, \tag{42}
$$

where ϕ_1, ϕ_2 and ϕ_3 are functions of $\mathscr{L}^{-1}\left\{\frac{(1-\omega)(p^{\omega}+\gamma_{\omega})}{\mathrm{B}(\omega)p^{\omega}}\mathscr{L}\right\}$.

Similarly, we can get

$$
\begin{aligned}
&\|\mathbb{K}(\mathbf{v}_{\mathfrak{q}}(t))-\mathbb{K}(\mathbf{v}_{\mathfrak{p}}(t))\| \leq (1+h(\mathscr{D})\phi_2(v)-\mu_{\mathrm{c}}\phi_3(v))\|(\mathbf{v}_{\mathfrak{q}}-\mathbf{v}_p)\|,\\
&\|\mathbb{K}(\mathbf{w}_{\mathfrak{q}}(t))-\mathbb{K}(\mathbf{w}_p(t))\| \leq (1-\varpi\mathbf{w}(0)\phi_4(v))\|(\mathbf{w}_{\mathfrak{q}}-\mathbf{w}_p)\|,
\end{aligned} \tag{43}
$$

where

$$
\begin{aligned}
&(1+q(\mathscr{D})(\mathrm{N}_1+\mathrm{N}_3)\phi_1(v)-h(\mathscr{D})\phi_2(v))<1,\\
&(1+h(\mathscr{D})\phi_2(v)-\mu_{\mathrm{c}}\phi_3(v))<1,\\
&(1-\varpi\mathbf{w}(0)\phi_4(v))<1.
\end{aligned}
$$

Consequently, $\mathbb{K}$ possesses a fixed point. Based on equations (42) and (43), we make the assumption that $\pi=(0,\ 0,\ 0)$

$$
\Lambda=\begin{cases}
(1+q(\mathscr{D})(\mathrm{N}_1+\mathrm{N}_3)\phi_1(v)-h(\mathscr{D})\phi_2(v)),\\
(1+h(\mathscr{D})\phi_2(v)-\mu_{\mathrm{c}}\phi_3(v)),\\
(1-\varpi\mathbf{w}(0)\phi_4(v)).
\end{cases}
$$

The criteria of Theorem (4.1) have been fulfilled. The proof is now complete.

Theorem 4.3 The special solution to Eq. (12) is generated uniquely using the iteration method.

Proof. The Hilbert space $\mathbb{F}: \ \mathrm{L}^2(m,\ n) \times (0,\ \ \mathrm{T})$ can be characterized as follows:

$$f: \ (m,\ n) \times (0,\ \ \mathrm{T}) \rightarrow, \quad \iint \quad d \quad d < \infty$$

When we consider the operator given, we have $\pi = (0,\ 0,\ 0)$.

$$\Lambda = \begin{cases} (1 + q(\mathscr{D})\,(\mathrm{N}_1 + \mathrm{N}_3)\,\phi_1(v) - h(\mathscr{D})\phi_2(v)), \\ \\ (1 + h(\mathscr{D})\phi_2(v) - \mu_{\mathrm{c}}\phi_3(v)), \\ \\ (1 - \varpi\mathbf{w}(0)\phi_4(v)). \end{cases}$$

By using

$$\mathbb{P}\left((\mathbf{u}_{11} - \mathbf{u}_{12},\ \mathbf{v}_{21} - \mathbf{v}_{22},\ \mathbf{w}_{31} - \mathbf{w}_{32}),\ (\Xi_1,\ \Xi_2,\ \Xi_3)\right).$$

We now have

$$\{(\mathbf{w}_{31} - \mathbf{w}_{32})\,q(\mathscr{D})\,(\mathbf{u}_{11} - \mathbf{u}_{12}) - h(\mathscr{D})\,(\mathbf{u}_{11} - \mathbf{u}_{12})\} \leq q(\mathscr{D})\,\|\mathbf{w}_{31} - \mathbf{w}_{32}\|\,\|\mathbf{u}_{11} - \mathbf{u}_{12}\|\,\|\Xi_1\|$$

$$+ h(\mathscr{D})\|\mathbf{u}_{11} - \mathbf{u}_{12}\|\|\Xi_1\|,$$

$$\{h(\mathscr{D})\,(\mathbf{u}_{11} - \mathbf{u}_{12}) - \mu_{\mathrm{c}}\,(\mathbf{v}_{21} - \mathbf{v}_{22})\} \leq h(\mathscr{D})\,\|\mathbf{u}_{11} - \mathbf{u}_{12}\|\,\|\Xi_2\| + \mu_{\mathrm{c}}\,\|\mathbf{v}_{11} - \mathbf{v}_{12}\|\,\|\Xi_2\|,$$

$$\{-\varpi\mathbf{w}(0)\,(\mathbf{w}_{31} - \mathbf{w}_{32})\} \leq \varpi\mathbf{w}(0)\,\|\mathbf{w}_{31} - \mathbf{w}_{32}\|\,\|\Xi_3\|.$$

For convergence solution, we have

$$\|\mathbf{u} - \mathbf{u}_{11}\|,\ \|\mathbf{u} - \mathbf{u}_{12}\| \leq \frac{\xi_{\mathrm{e}1}}{\varrho},$$

$$\|\mathbf{v} - \mathbf{v}_{21}\|,\ \|\mathbf{v} - \mathbf{v}_{22}\| \leq \frac{\xi_{\mathrm{e}2}}{\sigma},$$

$$\|\mathbf{w} - \mathbf{w}_{31}\|,\ \|\mathbf{w} - \mathbf{w}_{32}\| \leq \frac{\xi_e 3}{\tau}.$$

where

$$\varrho = 3\left(q(\mathscr{D})\left\|\mathbf{w}_{31}-\mathbf{w}_{32}\right\|\left\|\mathbf{u}_{11}-\mathbf{u}_{12}\right\|+h(\mathscr{D})\left\|\mathbf{u}_{11}-\mathbf{u}_{12}\right\|\right)\left\|\Xi_1\right\|,$$

$$\sigma = 3\left(h(\mathscr{D})\left\|\mathbf{u}_{11}-\mathbf{u}_{12}\right\|+\mu_c\left\|\mathbf{v}_{21}-\mathbf{v}_{22}\right\|\right)\left\|\Xi_2\right\|,$$

$$\tau = 3\left(\varpi\mathbf{w}(0)\left\|\mathbf{w}_{31}-\mathbf{w}_{32}\right\|\right)\left\|\Xi_3\right\|.$$

But it is obvious that

$$\left(q(\mathscr{D})\left\|\mathbf{w}_{31}-\mathbf{w}_{32}\right\|\left\|\mathbf{u}_{11}-\mathbf{u}_{12}\right\|+h(\mathscr{D})\left\|\mathbf{u}_{11}-\mathbf{u}_{12}\right\|\right)\neq 0,$$

$$\left(h(\mathscr{D})\left\|\mathbf{u}_{11}-\mathbf{u}_{12}\right\|+\mu_c\left\|\mathbf{v}_{21}-\mathbf{v}_{22}\right\|\right)\neq 0,$$

$$\left(\varpi\mathbf{w}(0)\left\|\mathbf{w}_{31}-\mathbf{w}_{32}\right\|\right)\neq 0,$$

where $\left\|\Xi_1\right\|,\ \left\|\Xi_2\right\|,\ \left\|\Xi_3\right\|\neq 0$. Therefore, we have

$$\left\|\mathbf{u}_{11}-\mathbf{u}_{12}\right\|=0,\ \left\|\mathbf{v}_{21}-\mathbf{v}_{22}\right\|=0,\ \left\|\mathbf{w}_{31}-\mathbf{w}_{32}\right\|=0.$$

Which yields that

$$\mathbf{u}_{11}=\mathbf{u}_{12},\ \mathbf{v}_{21}=\mathbf{v}_{22},\ \mathbf{w}_{31}=\mathbf{w}_{32}.$$

We achieve the desired outcome. Therefore, it is proven.

5. Numerical results and discussion

In this study, we analyze a numerical simulation of the tumor growth model using the MABC technique. We employ an innovative methodology for tumor growth by utilizing the fractional operator. The integral order derivative specifically analyzes tumor growth at a single location, while the fractional order derivative studies tumor growth from the initial point of infection to the final point of tumor growth. Examining the entire progression of tumor growth from its initiation to its end is indeed beneficial. The initial circumstances and parameter values for the simulation are provided in the following list [14]:

$$\mathbf{w}(0)=0.6,\ \mu_c=0.4,\ \varpi=0.8,\ q(\mathscr{D})=0.4,\ h(\mathscr{D})=0.6,\ \mathscr{D}=20.$$

$$\mathbf{u}(0)=10,\ \mathbf{v}(0)=0,\ \mathbf{w}(0)=0.4.$$

We use the iterative Laplace transform method (ILTM) for up to four terms in a row to get close to the answer of the fractional tumor growth model (12). We utilize the MABC fractional derivative to generate numerical results for the model at various fractional values around the stable state point. When exposed to irradiation, tumor growth decreases. During treatment, the volume of a growing tumor may fluctuate, either decreasing or increasing, but ultimately decreases after treatment. The "volume of non-dividing cells" undergoes a non-stationary process during therapy and subsequently decreases post-treatment. The rate of tumor growth decreases throughout the course of treatment. "Volume of proliferating tumor", "volume of non-dividing cells", and "tumor growth rate" decrease with the positive effect of radiation. The modified ABC fractional operator has proven to be highly accurate and efficient in estimating approximations of solutions for mathematical models of infectious illnesses. The simulation clearly illustrates that the use of fractional derivatives, as opposed to classical derivatives, leads to a more accurate approximation for controlling the disease.

6. Convergence analysis

The series (31-33) is a uniformly convergent solution to the exact solution. We utilize methodologies to ascertain the convergence of series (31-33), as referenced in [2]. Based on [2], we present the following theorem that establishes the convergence of this approach under certain condition:

Theorem 6.1 Suppose Ψ represent "Banach Space" and $\Pi : \Psi \to \Psi$ represent a "contractive nonlinear operator". In this case, there exists

$$X,\ X' \in \Psi,\ \left\|\Psi(X) - \Psi\left(X'\right)\right\| \leq d\left\|X - X'\right\|,\ 0 < d < 1.$$

By applying the Banach contracting principal, it can be proven that $\exists$ a unique point X, represented as $X = (\mathbf{u},\ \mathbf{v},\ \mathbf{w})$, s.t. $\Pi X = X$. The adomian-decomposition method (ADM) can be employed to express the series presented in equations (31-33) in the form that follows:

$$X_\ell = TX_{\ell-1},\ X_{\ell-1} = \sum_{j=0}^{\ell-1} X_j,\ \ell = 1,\ 2,\ 3\ \ldots$$

suppose that $X_0 \in \xi_r(X)$, where $\xi_r(X) = X' \in \Psi : \|X - X'\| < r$; then we have

(1) $X_\ell \in \xi_r(X)$,

(2) $\lim_{\ell \to \infty} X_\ell = X$.

Proof. (1) To establish this, we will use mathematical induction. When $\ell = 1$, we get

$$\|X_0 - X\| = \|\Pi(X_0) - \Pi(X)\| \leq d\|X_0 - X\|.$$

Given that the result is true for $\ell - 1$, we may assume the following:

$$\|X_0 - X\| \leq d^{\ell-1}\|X_0 - X\|.$$

We get

$$\|X_\ell - X\| = \|\Pi(X_{\ell-1}) - \Pi(X)\| \leq d\|X_{\ell-1} - X\| \leq d^\ell \|X_0 - X\|,$$

$$\|X_\ell - X\| = d^\ell \|X_0 - X\| \leq d^\ell r \leq r,$$

$$\Rightarrow X_\ell \in \xi_r(X).$$

(2) Since $\|X_\ell - X\| \leq d^\ell \|X_0 - X\|$ and $\lim_{\ell \to \infty} d^\ell = 0$, herefore, we have the

$$\lim_{\ell \to \infty} \|X_\ell - X\| = 0,$$

$$\Rightarrow \lim_{\ell \to \infty} X_\ell = X.$$

7. Conclusion

This research presents a novel MABC-fractional order model for simulating the growth of tumor epidemics. We developed a mathematical model of fractional order to equip healthcare providers with effective control techniques for managing this infectious disease in the broader population. The present study has demonstrated that a modified Atangana-Baleanu Caputo fractional derivative is capable of modeling infectious disorders. We have shown that the MABC-fractional tumor development model has close solutions, and we can display the results using the iterative Laplace transform method. We have employed Banach's fixed point theorem to validate the stability criteria and the existence of steady solutions. Numerical simulations study the actual behavior of the model. In mathematical models of infectious diseases, the memory parts of the MABC derivative look into the unseen dynamics of infection. Integer-order derivatives cannot perform this function. When we provide exact calculations regarding the communication structure in real-time, this model demonstrates high reliability. In terms of modeling, the proposed adjustment will offer valuable insight into the presence or absence of singularities at the origin. By employing the MABC derivative, we may more effectively describe the dynamics of intricate processes. This approach will significantly broaden the scope of scenarios where we can apply these operators.

Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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