Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

On q Sturm Liouville Operator with Periodic Boundary Conditions

Olgun Cabrİ[1], **Suayip Toprakseven**[2*]

[1] Department of Business Management, Artvin Coruh University, Artvin, Turkey
[2] Department of Accounting and Tax, Artvin Coruh University, Artvin, Turkey
E-mail: topraksp@artvin.edu.tr

Received: 27 June 2024; **Revised:** 27 September 2024; **Accepted:** 10 October 2024

Abstract: In this study, we consider q-Sturm Liouville operator with periodic boundary conditions. An asymptotic expression of the solution is obtained. With the help of this asymptotic representation, an asymptotic solution of the characteristic equation is presented. An application of the Rouche theorem, asymptotic expressions of eigenvalues are obtained.

***Keywords*:** periodic boundary conditions, q-Sturm Liouville, eigenvalues, eigenfunctions.

MSC: 39A13, 34B05, 34B24, 34L15, 47B50

1. Introduction

In this paper, we shall consider the following q-analogue of the Sturm-Liouville problem involving q-differential operator

$$l(y) := \frac{-1}{\mathrm{q}} D_{\mathrm{q}^{-1}} D_{\mathrm{q}} y(x) + r(x)y(x) = \lambda y(x),\ x \in (0,\ b), \tag{1}$$

subject to the periodic boundary conditions

$$V_1(y) = y(b) - y(0) = 0, \tag{2}$$

$$V_2(y) = D_{\mathrm{q}^{-1}} y(b) - D_{\mathrm{q}^{-1}} y(0) = 0, \tag{3}$$

where $\lambda \in \mathbb{C}$ is called an eigenvalue, $r(x) \in C([0,\ b))$ is real-valued function, $0 < \mathrm{q} < 1$ is real number, $b > 0$ and D_{q} denotes the q-derivative of a function f given by

DOI: https://doi.org/10.37256/cm.5420245208

$$D_{\mathrm{q}}f(x) = \frac{f(\mathrm{q}x) - f(x)}{(\mathrm{q}-1)x},\ x \in (0,\ b],$$

and q-derivative of function f at zero is defined by

$$D_{\mathrm{q}}f(0) = \lim_{n\to\infty} \frac{f(\mathrm{q}^n x) - f(0)}{\mathrm{q}^n x},\ x \in (0,\ b].$$

The q-difference or quantum calculus was initially developed by Jackson in the early 20th century [1, 2]. Most of the basic facts of quantum calculus such as q-derivative, q-Taylor formula, q-anti-derivative, Jackson integral and q-trigonometric functions have been studied in [3]. q-difference equations have been widely used in various fields, such as mathematical physics including heat and wave equations, dynamical systems, quantum models, sampling theory of signal analysis [4–11].

In recent years, many researchers have focused on q-Sturm-Liouville equations within the framework of classical Sturm-Liouville problems. In [12, 13], the authors studied the problem (1) with the separable boundary conditions

$$U_1(y) = a_{11}y(0) + a_{12}D_{\mathrm{q}^{-1}}y(0) = 0,$$

$$U_2(y) = b_{11}y(a) + a_{12}D_{\mathrm{q}^{-1}}y(a) = 0.$$

They have derived an asymptotic expression of eigenvalues and eigenfunctions. They found that $m \to \infty$,

$$\sqrt{\lambda_m} = \begin{cases} \dfrac{q^{-m+1/2}}{a(1-q)}\left(1 + O\left(q^{m/2}\right)\right), & a_{12} \neq 0, \\[2ex] \dfrac{q^{-m+1}}{a(1-q)}\left(1 + O\left(q^{m/2}\right)\right), & a_{12} = 0. \end{cases}$$

In [14, 15], the authors studied the continuous spectrum of singular q-Sturm-Liouville operators, and they established specific criteria for determining the limit-point character of the q-Sturm-Liouville equation at infinity. In [16], the authors examined the q-Sturm-Liouville problem containing a discontinuity condition at an interior point of the domain. In particular, q-difference equations have been frequently used in physics problems such as dynamical systems and quantum models [17], for q-derivative versions of physics phenomenons such as heat and wave equations [18], and signal analysis [19, 20]. For the detailed discussions, the reader can be referred to [21].

In [22], the author investigated the existence of positive solutions to the nonlinear q-fractional boundary value problem. In [23], the author studied the existence of a singular Hahn difference equation of a q, ω-Sturm-Liouville problem with transmission conditions.

In the present work we will focus on the q-Sturm-Liouville problem (1) endowed with the periodic boundary conditions. Our boundary conditions are non-separable and we shall obtain an asymptotic expression of eigenvalues of the problem (1)-(3).

This paper is organized as follows. In Section 2, we introduce the notations and some preliminaries. In Section 3, we derive the asymptotic expansion of the solution and examine the eigenvalues of the problems. In Section 4, an example has been given to validate the analysis. In Section 5, we close the paper with some conclusion remarks.

2. Notations and preliminary definitions

In this section, we introduce some q-notations and symbols that will be utilized throughout the paper.

Definition 1 For $\omega \in \mathbb{C}$ and $m \in \mathbb{N}$, we define the q-shifted factorial (see, e.g., [24, 25])

$$(\omega;\ \mathfrak{q})_0 = 1,\ (\omega;\ \mathfrak{q})_m = \prod_{i=0}^{m-1}(1-\omega\mathfrak{q}^i),\ (\omega;\ \mathfrak{q})_\infty = \prod_{i=0}^{\infty}(1-\omega\mathfrak{q}^i).$$

Definition 2 [25] The generalized form of the q-shifted factorial is given as follows:

$$(\omega;\ \mathfrak{q})_\nu = \frac{(\omega;\ \mathfrak{q})_\infty}{(\omega\mathfrak{q}^\nu;\ \mathfrak{q})_\infty},\ (\nu \in \mathbb{R}). \tag{4}$$

Definition 3 [25] The θ-function is described as follows:

$$\theta(\omega;\ \mathfrak{q}) = \sum_{r=-\infty}^{\infty} \mathfrak{q}^{r^2}\omega^r,\ \omega \in \mathbb{C}. \tag{5}$$

For $\nu > -1$, the third type of q-Bessel functions is expressed as

$$J_\nu(\omega;\ \mathfrak{q}) = \omega^\nu \frac{(\mathfrak{q}^{\nu+1};\ \mathfrak{q})_\infty}{(\mathfrak{q};\ \mathfrak{q})_\infty} \sum_{r=0}^{\infty}(-1)^r \frac{\mathfrak{q}^{r(r+1)/\omega}}{(\mathfrak{q};\ \mathfrak{q})_r(\mathfrak{q}^{\nu+1};\ \mathfrak{q})_r}\omega^{2r},\ \omega \in \mathbb{C}.$$

Definition 4 [25] The fundamental trigonometric functions $\cos(\omega;\ \mathfrak{q})$ and $\sin(\omega;\ \mathfrak{q})$ are specified on $\mathbb{C}$

$$\cos(\omega,\ \mathfrak{q}) = \sum_{r=0}^{\infty}(-1)^r \frac{\mathfrak{q}^{r^2}(1-\mathfrak{q})^{2r}\omega^{2r+1}}{(\mathfrak{q},\ \mathfrak{q})_{2r}}$$

$$= \frac{(\mathfrak{q}^2;\ \mathfrak{q}^2)_\infty}{(\mathfrak{q};\ \mathfrak{q}^2)_\infty}(\omega\mathfrak{q}^{-1/2}(1-\mathfrak{q}))^{1/2}J_{-1/2}(\omega(1-\mathfrak{q})/\sqrt{\mathfrak{q}};\ \mathfrak{q}^2),$$

$$\sin(\omega,\ \mathfrak{q}) = \sum_{r=0}^{\infty}(-1)^r \frac{\mathfrak{q}^{r(r+1)}(1-\mathfrak{q})^{2r+1}\omega^{2r+1}}{(\mathfrak{q},\ \mathfrak{q})_{2r+1}}$$

$$= \frac{(\mathfrak{q}^2;\ \mathfrak{q}^2)_\infty}{(\mathfrak{q};\ \mathfrak{q}^2)_\infty}(\omega\mathfrak{q}^{1/2}(1-\mathfrak{q}))^{1/2}J_{-1/2}(\omega(1-\mathfrak{q});\ \mathfrak{q}^2).$$

Definition 5 [25] The Hilbert space $L^2_{\mathfrak{q}}(0,\ b)$ is defined by

$$L^2_{\mathfrak{q}}(0,\ b) = \{f : [0,\ b] \to \mathbb{C} : \int_0^b |f(t)|^2 d_{\mathfrak{q}}t < \infty\}, \tag{6}$$

where the Jackson integration is given by [2]

$$\int_0^x f(t)d_{\mathfrak{q}}t = x(1-\mathfrak{q})\sum_{r=0}^{\infty}\mathfrak{q}^r f(\mathfrak{q}^r x),\ x \geq 0. \tag{7}$$

Definition 6 [26] Assume that $0 < |\mathfrak{q}| < 1$ and that $\eta \in \mathbb{C}, |\eta| \geq 1$. For $r \in \mathbb{Z}^+$, let $\mathfrak{A}_r$ be the annulus given by

$$\mathfrak{A}_r = \left\{\eta \in \mathbb{C}:\ \mathfrak{q}^{-2r+2} \leq |\eta| \leq \mathfrak{q}^{-2r}\right\}. \tag{8}$$

Thus, as $r \to \infty$, the following inequality holds:

$$\log\theta(\eta;\ \mathfrak{q}) \leq \frac{-(\log(|\eta|^2)}{4\log\mathfrak{q}} + \log|1+\mathfrak{q}^{2r-1}\eta| + \mathscr{O}(1). \tag{9}$$

Definition 7 Let $g(\omega) = \sum_{n=0}^{\infty} a_n\omega^n$ be an entire function. denoted by of g, $\rho(g)$ is given by

$$\rho(g) = \lim_{n\to\infty}\sup\frac{n\log n}{\log|a_n|^{-1}}. \tag{10}$$

Definition 8 [26] The zeros of $J_\nu(.,\ \mathfrak{q}^2)$, with $\nu > -1$ are real and has a countably infinite number of positive simple zeros.

3. Asymptotic expression of solutions and eigenvalues

3.1 *Asymptotic expression of solutions*

In [13], the following asymptotic expression for the solution of (1) has been proved

$$\psi_1(\omega,\ \mu) = \cos(\mu\omega;\ \mathfrak{q}) + O\left(\frac{e^{\frac{-(\log|\mu|\omega(1-\mathfrak{q}))^2}{\sqrt{\mathfrak{q}}\log\mathfrak{q}}}}{|\mu|}\right), \tag{11}$$

$$\psi_2(\omega,\ \mu) = \frac{\sin(\mu\omega;\ \mathfrak{q})}{\mu} + O\left(\frac{e^{\frac{-(\log|\mu|\omega(1-\mathfrak{q}))^2}{\sqrt{\mathfrak{q}}\log\mathfrak{q}}}}{|\mu|^2}\right), \tag{12}$$

$$D_{\mathfrak{q}^{-1}}\psi_1(\omega,\ \mu) = -\sqrt{\mathfrak{q}}\sin(\mu\sqrt{\mathfrak{q}}\omega;\ \mathfrak{q}) + O\left(e^{\frac{-(\log|\mu|\omega(1-\mathfrak{q}))^2}{\sqrt{\mathfrak{q}}\log\mathfrak{q}}}\right), \tag{13}$$

$$D_{\mathrm{q}^{-1}}\psi_2(\omega,\ \mu)=\cos(\mu\sqrt{\mathrm{q}}\omega;\ \mathrm{q})+O\left(\frac{e^{\dfrac{-(\log|\mu|\omega(1-\mathrm{q}))^2}{\sqrt{\mathrm{q}}\log\mathrm{q}}}}{|\mu|}\right), \tag{14}$$

where μ is complex number with $\mu=\sqrt{\lambda}$. Now we will give the following auxiliary lemma.

Lemma 9 Let $\mathrm{q}\in(0,\ 1)$ and $x\in(0,\ b]$. Then

$$\sqrt{\mathrm{q}}\sin(bx;\ \mathrm{q})\sin(\frac{bx}{\sqrt{\mathrm{q}}};\ \mathrm{q})+\cos(bx;\ \mathrm{q})\cos(\frac{bx}{\sqrt{\mathrm{q}}};\ \mathrm{q})=1.$$

Proof. Define the function f given by

$$f(x,\ \mathrm{q})=\sqrt{\mathrm{q}}\sin(bx;\ \mathrm{q})\sin(\frac{bx}{\sqrt{\mathrm{q}}};\ \mathrm{q})+\cos(bx;\ \mathrm{q})\cos(\frac{bx}{\sqrt{\mathrm{q}}};\ \mathrm{q}).$$

Clearly, one has $f(0,\ \mathrm{q})=1$. Let us prove that

$$D_{\mathrm{q}^{-1}}f(x,\ \mathrm{q})=0.$$

It is known from [3]

$$D_{q^{-1}}(u(x)v(x))=u(q^{-1}x).\ D_{q^{-1}}v(x)+D_{q^{-1}}u(x).\ v(x).$$

Using $D_{q^{-1}}\sin(bx;\ \mathrm{q})=b\cos(b\sqrt{\mathrm{q}x})$ and $D_{q^{-1}}\cos(bx;\ \mathrm{q})=\sqrt{\mathrm{q}}b\sin(b\sqrt{\mathrm{q}x})$, one can check that $D_{\mathrm{q}^{-1}}f(x,\ \mathrm{q})=0$. This implies that $f(\mathrm{q}^{-1}x,\ \mathrm{q})=f(x,\ \mathrm{q})$. In addition, $f(x,\ \mathrm{q})$ has formal power series, then we have $\mathrm{q}^n c_n=c_n$ for each n, where c_n is the coefficient of x^n. This is possible only when $c_n=0$ for any $n\geq 1$, i.e., $f(x,\ \mathrm{q})$ is constant. Thus, $f(x,\ \mathrm{q})=1$. The proof is completed. □

3.2 *Asymptotic expression of characteristic determinant*

Theorem 10 As $|\lambda|\to\infty$, the characteristic determinant of the problem (1)-(3) has the following form

$$\Delta(\mu)=\sum_{r=1}^{\infty}(-1)^r\frac{\mathrm{q}^{r^2-r}(1+\mathrm{q}^r)(1-\mathrm{q})^{2r}(b\mu)^{2r}}{(\mathrm{q},\ \mathrm{q})_{2r}}+O\left(e^{\left(\dfrac{-(\log|\mu|b(1-\mathrm{q}))^2}{\sqrt{\mathrm{q}}\log\mathrm{q}}\right)}\right). \tag{15}$$

Proof. Using (11)-(14) and Lemma 3.1, we have

$$\Delta(\mu) = \cos(b\mu;\ \mathfrak{q}) + \cos(\frac{b\mu}{\sqrt{\mathfrak{q}}};\ \mathfrak{q}) - 1 - \sqrt{\mathfrak{q}}\sin(b\mu;\ \mathfrak{q})\sin(\frac{b\mu}{\sqrt{\mathfrak{q}}};\ \mathfrak{q})$$

$$-\cos(b\mu;\ \mathfrak{q})\cos(\frac{b\mu}{\sqrt{\mathfrak{q}}};\ \mathfrak{q}) + O\left(e^{\frac{-(\log|\mu|b\mathfrak{q}^{-1/2}(1-\mathfrak{q}))^2}{\sqrt{\mathfrak{q}}\log\mathfrak{q}}}\right)$$

$$= \cos(b\mu;\ \mathfrak{q}) + \cos(b\mu;\ \mathfrak{q}) - 2 + O\left(e^{\frac{-(\log|\mu|b(1-\mathfrak{q}))^2}{\sqrt{\mathfrak{q}}\log\mathfrak{q}}}\right).$$

Using the expansion series of $cos(x;\ \mathfrak{q})$, we have

$$\Delta(\mu) = \sum_{r=1}^{\infty}(-1)^r\frac{\mathfrak{q}^{r^2-r}(1+\mathfrak{q}^r)(1-\mathfrak{q})^{2r}(b\mu)^{2r}}{(\mathfrak{q},\ \mathfrak{q})_{2r}} + O\left(e^{\frac{-(\log|\mu|b(1-\mathfrak{q}))^2}{\sqrt{\mathfrak{q}}\log\mathfrak{q}}}\right). \tag{16}$$

This ends the proof. □

Before obtaining the zeros $\Delta(\mu)$, we will obtain an asymptotic expression for the roots of the series given by

$$F(\mu) = \sum_{r=1}^{\infty}(-1)^r\frac{\mu^{r^2-r}(1+\mu^r)(1-\mu)^{2r}(\mu)^{2r}}{(\mu,\mu)_{2r}},\ \mu\in(0,\ 1). \tag{17}$$

Lemma 11 The positive zeros of $F(\mu)$ are given by

$$x_r = \frac{\mathfrak{q}^{1-r}}{(1-\mathfrak{q})}(1+O(q^{r/2})),\ \text{as } r\to\infty. \tag{18}$$

Proof. Let $\sigma := (\mathfrak{q},\ \mathfrak{q})_\infty$ and define the function $H(\mu)$ given as

$$H(\mu) = F(\mu) - \frac{1}{\sigma}\theta(\frac{-(\mu(1-\mathfrak{q}))^2}{\mathfrak{q}})$$

$$= \sum_{r=-\infty}^{+\infty}\delta_{2r}\mu^{2r},$$

where

$$\delta_{2r} = \begin{cases} (-1)^r \dfrac{\mathfrak{q}^{r^2-r}((1-\mathfrak{q}))^{2r}}{\sigma}\left[(1+\mathfrak{q}^r)(\mathfrak{q}^{2r+1},\ \mathfrak{q})-1\right], & r \geq 1, \\ (-1)^r \dfrac{\mathfrak{q}^{r^2-r}((1-\mathfrak{q}))^{2r}}{\sigma}, & r < 1. \end{cases} \tag{19}$$

From the expansion of series (4) for $r \geq 1$, we have

$$| (1+\mathfrak{q}^r)(\mathfrak{q}^{2r+1},\ \mathfrak{q}) - 1 | \leq | \mathfrak{q}^r | + | \mathfrak{q}^r + 1 || (\mathfrak{q}^{2r+1},\ \mathfrak{q}) |$$

$$\leq \mathfrak{q}^r + \frac{\mathfrak{q}^{r+1}\mathfrak{q}^{2r+1}}{(1-\sqrt{\mathfrak{q}})\sigma} \leq \frac{2\mathfrak{q}^r}{(1-\sqrt{\mathfrak{q}})\sigma}.$$

Thus, we get

$$| \delta_{2r} | \leq \frac{2\mathfrak{q}^{r^2}((1-\mathfrak{q})^2)}{(1-\sqrt{\mathfrak{q}})\sigma^2}. \tag{20}$$

Hence, for $\mu \in \mathbb{C} - \{0\}$ we arrive at

$$| H(\mu) | \leq \frac{2}{(1-\sqrt{\mathfrak{q}})\sigma^2}\theta((|\mu|(1-\mathfrak{q}))^2;\ \mathfrak{q}). \tag{21}$$

By taking the logarithm of both sides of (21) for $\mu \in \mathbb{C} - \{0\}$, we obtain at once

$$\log|H(\mu)| \leq \log\theta((|\mu|(1-\mathfrak{q}))^2;\ \mathfrak{q}) + \log\frac{2}{(1-\sqrt{\mathfrak{q}})\sigma^2}. \tag{22}$$

Theorem 6 tells us that if $\mu^2(1-\mathfrak{q})^2 \in \mathfrak{A}_r$, $r \geq r_0$ and

$$\mathfrak{q}^{-2r+2} \leq (1-\mathfrak{q})^2|\mu|^2 \leq \mathfrak{q}^{-2r}, \tag{23}$$

then, we obtain

$$\log\theta((|\mu|(1-\mathfrak{q}))^2;\ \mathfrak{q}) \le \frac{-(\log(|\mu|(1-\mathfrak{q}))^2)^2}{4\log\mathfrak{q}} + \log|1+\mathfrak{q}^{2r-1}(|\mu|(1-\mathfrak{q}))^2| + \mathscr{O}(1)$$

$$\le \frac{-(\log(|\mu|(1-\mathfrak{q}))^2)^2}{4\log\mathfrak{q}} + \mathfrak{q}^{-1} + \mathscr{O}(1)$$

$$= \frac{-(\log(|\mu|(1-\mathfrak{q})))^2}{\log\mathfrak{q}} + \mathscr{O}(1)$$

$$= \frac{-(\log(|\mu|(1-\mathfrak{q})\mathfrak{q}^{-1/2}))^2}{\log\mathfrak{q}} - \log|\mu| - \log(1-\mathfrak{q})\mathfrak{q}^{-1/2} - \log\mathfrak{q}^{1/2} + \mathscr{O}(1).$$

For $r > r_0$, we define the annulus $\tilde{\mathfrak{A}}_r$ given by

$$\tilde{\mathfrak{A}}_r = \left\{\mu \in \mathbb{C}:\ \mathfrak{q}^{-2r+2} \le \frac{((1-\mathfrak{q})|\mu|)^2}{\mathfrak{q}} \le \mathfrak{q}^{-2r}\right\}. \tag{24}$$

Then, we obtain

$$\frac{-\log(\dfrac{(|\mu|(1-\mathfrak{q}))^2}{\mathfrak{q}})^2}{4\log\mathfrak{q}} + \log|1-\mathfrak{q}^{2r-2}(\mu(1-\mathfrak{q}))^2| + C \le \log\theta(\frac{-(|\mu|(1-\mathfrak{q}))^2}{\mathfrak{q}};\ \mathfrak{q}) + \mathscr{O}(1). \tag{25}$$

Therefore, we deduce that

$$\frac{-(\log(\dfrac{|\mu|(1-\mathfrak{q})}{\sqrt{\mathfrak{q}}}))^2}{\log\mathfrak{q}} \le \log\theta(\frac{-(|\mu|(1-\mathfrak{q}))^2}{\mathfrak{q}};\ \mathfrak{q}) - \log|1-\mathfrak{q}^{2r-2}(\mu(1-\mathfrak{q}))^2| + \mathscr{O}(1). \tag{26}$$

Hence, for $\mu \in \tilde{A}_r$ one has

$$\log|H(\mu)| \le \log\theta(\frac{-(|\mu|(1-\mathfrak{q}))^2}{\mathfrak{q}};\ \mathfrak{q}) - \log|1-\mathfrak{q}^{2r-2}(\mu(1-\mathfrak{q}))^2| - \log|\mu| + C_1, \tag{27}$$

where $C_1 = -\log(1-\mathfrak{q})\mathfrak{q}^{-1/2} - \log\mathfrak{q}^{-1/2} + \mathscr{O}(1)$. This implies that if $\mu \in \partial\tilde{\mathfrak{A}}_r$ we have

$$|1-\mathfrak{q}^{2r-2}(\mu(1-\mathfrak{q}))^2| \ge \mathfrak{q}^{2r-2}(|\mu|(1-\mathfrak{q}))^2 - 1 = \mathfrak{q}^{-1}(1-\mathfrak{q});\quad |\mu| = \mathfrak{q}^{-r+1/2},$$

$$|1-\mathfrak{q}^{2r-2}(\mu(1-\mathfrak{q}))^2| \ge 1 - \mathfrak{q}^{2r-2}(|\mu|(1-\mathfrak{q}))^2 = (1-\mathfrak{q});\quad |\mu| = \mathfrak{q}^{-r+3/2}.$$

This is equivalent to write

$$\log|1-\mathfrak{q}^{2r-2}(\mu(1-\mathfrak{q}))^2| \geq \log(1-\mathfrak{q}), \quad \mu \in \partial\tilde{\mathfrak{A}}_r. \tag{28}$$

Consequently, for any μ on the boundary of $\tilde{\mathfrak{A}}_r$ we have

$$\log|H(\mu)| \leq \log\theta(\frac{-(|\mu|(1-\mathfrak{q}))^2}{\mathfrak{q}};\ \mathfrak{q}) - \log((1-\mathfrak{q})|\mu|) + C_1.$$

If we choose r_0 such that $-\log((1-\mathfrak{q})|\mu|) + C_1 < 0$, then the following inequality holds:

$$|H(\mu)| \leq |\theta(\frac{-(|\mu|(1-\mathfrak{q}))^2}{\mathfrak{q}};\ \mathfrak{q})|, \quad \mu \in \partial\tilde{\mathfrak{A}}_r$$

$$= K(\mu).$$

Applying the Rouché theorem, we deduce that the number of zeros of the function $H(\mu)$ and $H(\mu)+K(\mu) = F(\mu)$ are the same in $\tilde{\mathfrak{A}}_r$. Thus, two simple roots of the function$F(\mu)$ are real and symmetric in $\tilde{\mathfrak{A}}_r$. Suppose that $x_r, r > r_0$ is the positive zeros of $F(\mu)$. Then one can see that

$$\log|1-\mathfrak{q}^{2r-2}(x_r(1-\mathfrak{q}))^2| \leq -\log x_r + C_1 \Rightarrow 2\log|1-\mathfrak{q}^{r-1}x_n(1-\mathfrak{q})| \leq -\log x_r + C_1$$

$$\Rightarrow \log|1-\mathfrak{q}^{n-1}x_r(1-\mathfrak{q})| \leq -\frac{1}{2}\log x_r + C_1,$$

which shows that $|1-\mathfrak{q}^{r-1}x_r(1-\mathfrak{q})| = O(x_r^{-1/2})$. We note that $\mathfrak{q}^{2r-2}(x_r(1-\mathfrak{q}))$ is positive, and the above inequality holds true. Then, we obtain

$$x_r = \frac{\mathfrak{q}^{1-r}}{1-\mathfrak{q}}(1+O(\mathfrak{q}^{r/2}) \text{ as } r \to \infty. \tag{29}$$

Hence, the proof is completed. □

Now we give the lemma about the asymptotic behavior of $|F(\omega)|$. For this purpose we define a sequence $\{\zeta_k\}_{k=1}^{\infty}$ given by

$$\zeta_k = \frac{\log|\frac{x_k}{x_{k+1}}|}{\log\alpha}. \tag{30}$$

Then, we see that

$$\zeta_k \to 1 \text{ as } k \to \infty.$$

Set $\zeta = \inf_{k\in Z^+} \zeta_k$ such that $0 < \zeta \leq 1$. We now define the sequences $\{c_k\}_{k=1}^{\infty}$ and $\{d_k\}_{n=1}^{\infty}$ given by

$$c_k := \begin{cases} \frac{\zeta_k+\zeta}{2}, & \zeta_k \neq \zeta, \\ \frac{\zeta}{2} & \zeta_k = \zeta, \end{cases} \qquad d_1 = \frac{\zeta}{2}, \quad d_k = \begin{cases} \frac{\zeta_k-\zeta}{2} & \zeta_k \neq \zeta, \\ \frac{\zeta}{2} & \zeta_k = \zeta. \end{cases}$$

We note that

$$x_m \mathfrak{q}^{-c_n} = x_{m+1}\mathfrak{q}^{d_{m+1}}, \quad m \geq 1. \tag{31}$$

Through the division of the region $\left\{\omega \in C: \ |\omega| \geq \mathfrak{q}^{\zeta/2}x_1\right\}$, let us define the set

$$A_m^f = \left\{\omega \in C: \ x_m\mathfrak{q}^{d_m} \leq |\omega| \leq \mathfrak{q}^{-c_m}\right\} \quad m \geq 1. \tag{32}$$

The following lemma provides the asymptotic behavior of the function $|F(\omega)|$ will be used in the sequel.
Lemma 12 Assume that $|\omega| \geq \mathfrak{q}x_1$ then for $\omega \in A_m^f$,then

$$\log|F(\omega)| = \frac{-(\log(|\omega|(1-\mathfrak{q})))^2}{\log\mathfrak{q}} + \log|\frac{x_m^2-\omega^2}{x_m^2}| + \text{constant}, \quad m \to \infty \tag{33}$$

Proof. Since $F(\omega)$ is an entire function with zero order, the proof can be deduced from [27] using similar methodologies. □

3.3 *Asymptotic expression of eigenvalues*

Theorem 13 Asymptotic expression of the positive zeros λ_m of $\Delta(\mu)$ is, $m \to \infty$

$$\sqrt{\lambda_m} = \frac{\mathfrak{q}^{1-m}}{b(1-\mathfrak{q})}\left(1 + O(\mathfrak{q}^{m/2})\right), \tag{34}$$

Proof. We set $\Delta(\mu) = F(\mu) + S(\mu)$ where

$$S(\mu) = O\left(e^{\left(\frac{-(\log|\mu|b(1-\mathfrak{q}))^2}{\sqrt{\mathfrak{q}}\log\mathfrak{q}}\right)}\right), \tag{35}$$

Then there exists a positive constant c such that as $\mu \to \infty$

$$|S(\mu)| \leq ce^{\left(\frac{-(\log|\mu|b(1-\mathfrak{q}))^2}{\sqrt{\mathfrak{q}}\log \mathfrak{q}}\right)}. \tag{36}$$

Thus, from (33) if $\mu b \in A_m^f$, $m \geq 1$, we get, as $m \to \infty$

$$\log|F(\mu)| = \frac{-(\log(|\mu|b(1-q)))^2}{\log \mathfrak{q}} + \log|\frac{x_m^2 - b^2\mu^2}{x_m^2}| + \text{ constant.} \tag{37}$$

So there exists an integer l_0, and $\delta_1 > 0$ such that for $m \geq l_0$

$$\Big|\log F(\mu) + \frac{-(\log(|\mu|b(1-q)))^2}{\log \mathfrak{q}} - \log|\frac{x_m^2 - b^2\mu^2}{x_m^2}|\Big| \leq \delta_1. \tag{38}$$

Hence

$$\log|S(\mu)| \leq \log c - \frac{(\log|\mu|b(1-\mathfrak{q}))^2}{\sqrt{\mathfrak{q}}\log \mathfrak{q}}$$

$$\leq \log|F(\mu)| - \log\left|1 - \frac{b^2\mu^2}{x_m^2}\right| + \log c + \delta_1.$$

If $\omega \in \partial A_m^f$, then either $|\mu b| = x_m \mathfrak{q}^{dm}$

$$\log|1 - \frac{b^2\mu^2}{x_m^2}| \geq \log(1 - \mathfrak{q}^{2d_m}),\ |\mu b| = x_m \mathfrak{q}^{d_m} \tag{39}$$

or

$$\log|1 - \frac{b^2\mu^2}{x_m^2}| \geq \log(\mathfrak{q}^{2c_m - \mathfrak{q}}),\ |\mu b| = x_m \mathfrak{q}^{-c_m} \tag{40}$$

Since $\{c_m\}_{m=1}^{\infty}$ and $\{x_m\}_{m=1}^{\infty}$ are bounded, we can define

$$\delta_3 = inf_{m\geq 1} \log(1 - \mathfrak{q}^{2d_m}) \log(\mathfrak{q}^{-2c_m} - 1). \tag{41}$$

Thus, we obtain

$$\log|1-\frac{b^2\mu^2}{x_m^2}|\geq\delta_3. \tag{42}$$

By choosing l_0 sufficiently large, we can ensure that

$$|S(\mu)|\leq|F(\mu)|,\ \lambda\in\partial A_m^f.$$

Thus we get

$$\log|1-\frac{b^2\lambda_m}{x_m^2}|\leq-\log|\sqrt{\lambda_m}|+\delta_2.$$

This leads to write

$$\log|1-\frac{b\sqrt{\lambda_m}}{x_m}|\leq-\log|\lambda_m^{1/4}|+\frac{\delta_2}{2}. \tag{43}$$

Consequently, one has

$$\log\left|1-\frac{b\sqrt{\lambda}}{x_m}\right|=O(|\lambda_m^{-1/4}|),$$

$$\sqrt{\lambda_m}=b^{-1}x_m\left(1+O(\mathrm{q}^{m/2})\right),$$

$$\sqrt{\lambda_m}=\frac{\mathrm{q}^{1-m}}{b(1-\mathrm{q})}\left(1+O(\mathrm{q}^{m/2})\right).$$

This completes the proof. □

4. An example

Example 1. Consider the q-Sturm-Liouville boundary value problem with $b=1$ and $r(x)=0$

$$\frac{-1}{\mathrm{q}}D_{\mathrm{q}^{-1}}D_{\mathrm{q}}y(x)=\lambda y(x),\ x\in(0,1), \tag{44}$$

subject to the periodic boundary conditions

$$V_1(y) = y(1) - y(0) = 0, \tag{45}$$

$$V_2(y) = D_{q^{-1}}y(1) - D_{q^{-1}}y(0) = 0. \tag{46}$$

A fundamental set of the solutions is given by

$$\psi_1(x,\ \lambda) = \cos(\sqrt{\lambda}x;\ q),\ \psi_2(x,\ \lambda) = \frac{\sin(\sqrt{\lambda}x;\ q)}{\sqrt{\lambda}}.$$

The characteristic equation is given by

$$\Delta(\lambda) = 2 - \cos\left(\frac{\sqrt{\lambda}}{\sqrt{q}}\right) - \cos(\sqrt{\lambda}).$$

Hence, the eigenvalues $\{\lambda_r\}_{r=1}^{\infty}$ are the zeros of $\Delta(\mu)$. Hence, using Lemma 11 we can express the roots of $\Delta(\mu)$ asymptotically by

$$\sqrt{\lambda} = \frac{q^{1-r}}{1-q}(1 + O(q^{r/2})),\ \text{ as } r \to \infty. \tag{47}$$

5. Conclusion

In [25], the authors give the asymptotic expression for the problem (1) with separable boundary conditions. In this work, we examined the q-Sturm Liouville equation with periodic boundary conditions. Periodic boundary conditions are known as non-separable boundary conditions. By using asymptotic solution of solutions, we derived asymptotic solution of the characteristic equation. Finally, we presented the asymptotic expressions for the eigenvalues of the problem using the Rouche theorem.

Acknowledgment

The authors thank the anonymous referees for their constructive comments and careful checking of the manuscript, which improve the presentation.

Conflict of interest

The authors declare no potential conflicts of interest.

References

[1] Jackson FH. q-Difference equations. *American Journal of Mathematics*. 1910; 32(4): 305-314.

[2] Jackson FH. On q-definite integrals. *Quarterly Journal of Pure and Applied Mathematics*. 1910; 41: 193-203.
[3] Kac VG, Cheung P. *Quantum Calculus*. Springer; 2002.
[4] Abdel-Gawad H, Aldailami A. On q-dynamic equations modelling and complexity. *Applied Mathematical Modelling*. 2010; 34(3): 697-709.
[5] Field CM, Joshi N, Nijhoff FW. q-difference equations of KdV type and Chazy-type second-degree difference equations. *Journal of Physics A: Mathematical and Theoretical*. 2008; 41(33): 332005.
[6] Abdi WH. Certain inversion and representation formulae for q-Laplace transforms. *Mathematical Journal*. 1964; 83: 238-249.
[7] Abreu LD. Sampling theory associated with q-difference equations of the Sturm-Liouville type. *Journal of Physics A: Mathematical and General*. 2005; 38(48): 10311.
[8] Finkelstein RJ. The q-Coulomb problem. *Journal of Mathematical Physics*. 1996; 37(6): 2628-2636.
[9] Floreanini R, Vinet L. Automorphisms of the q-oscillator algebra and basic orthogonal polynomials. *Physics Letters A*. 1993; 180(6): 393-401.
[10] Floreanini R, Vinet L. Symmetries of the q-difference heat equation. *Letters in Mathematical Physics*. 1994; 32: 37-44.
[11] Floreanini R, Vinet L. Quantum symmetries of q-difference equations. *Journal of Mathematical Physics*. 1995; 36(6): 3134-3156.
[12] Annaby M, Mansour Z. Basic Sturm-Liouville problems. *Journal of Physics A: Mathematical and General*. 2005; 38(17): 3775.
[13] Annaby MH, Mansour ZS. Asymptotic formulae for eigenvalues and eigenfunctions of q-Sturm-Liouville problems. *Mathematische Nachrichten*. 2011; 284(4): 443-470.
[14] Allahverdiev BP, Tuna H. Qualitative spectral analysis of singular q-Sturm-Liouville operators. *Bulletin of the Malaysian Mathematical Sciences Society*. 2020; 43(2): 1391-1402.
[15] Allahverdiev B, Tuna H. Eigenfunction expansion in the singular case for q-Sturm-Liouville operators. *Caspian Journal of Mathematical Sciences (CJMS)*. 2019; 8(2): 91-102.
[16] Karahan D, Mamedov KR. Sampling theory associated with q-Sturm-Liouville operator with discontinuity conditions. *Journal of Contemporary Applied Mathematics*. 2020; 10(2): 1-9.
[17] Abdel-Gawad H, Aldailami A. On q-dynamic equations modelling and complexity. *Applied Mathematical Modelling*. 2010; 34(3): 697-709.
[18] Field CM, Joshi N, Nijhoff FW. q-difference equations of KdV type and Chazy-type second-degree difference equations. *Journal of Physics A: Mathematical and Theoretical*. 2008; 41(33): 332005.
[19] Abdi WH. Certain inversion and representation formulae for q-Laplace transforms. *Mathematical Journal*. 1964; 83: 238-249.
[20] Abreu LD. Sampling theory associated with q-difference equations of the Sturm-Liouville type. *Journal of Physics A: Mathematical and General*. 2005; 38(48): 10311.
[21] Ernst T. *ThE History of Q-Calculus and a New Method*. Citeseer; 2000.
[22] Ferreira RA. Positive solutions for a class of boundary value problems with fractional q-differences. *Computers & Mathematics with Applications*. 2011; 61(2): 367-373.
[23] Palamut Kosar N. On a spectral theory of singular Hahn difference equation of a Sturm-Liouville type problem with transmission conditions. *Mathematical Methods in the Applied Sciences*. 2023; 46(9): 11099-11111.
[24] Gasper G, Rahman M. *Basic Hypergeometric Series*. Cambridge university press; 2011.
[25] Jackson FH. q-difference equations. *q-Fractional Calculus and Equations*. 1910; 32(4): 305-314.
[26] Bergweiler W, Haymana WK. Zeros of solutions of a functional equation. *Computational Methods and Function Theory*. 2004; 3(1): 55-78.
[27] Annaby M, Mansour Z. A basic analog of a theorem of Pólya. *Mathematical Journal*. 2008; 258(2): 363-379.