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Research Article

Petrov-Galerkin Lucas Polynomials Procedure for the Time-Fractional Diffusion Equation

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Abstract: Herein, we build and implement a combination of Lucas polynomials basis that fulfills the spatial homogenous boundary conditions. This basis is then used to solve the time-fractional diffusion equation spectrally. The elements of all spectral matrices are explicitly obtained in terms of the Gauss hypergeometric function. The convergence and error analysis of the proposed Lucas expansion is studied. Numerical results indicate the high accuracy and applicability of the suggested algorithm.

***Keywords*:** time-fractional diffusion equation, Lucas polynomials, Lucas number, golden ratio, Petrov-Galerkin method, Convergence analysis

MSC: 65M60, 11B39, 40A05, 34A08

1. Introduction

Many models that describe many phenomena in the applied sciences can be modeled by Fractional Differential Equations (FDEs), see for example [1-5]. One of the fundamental equations of mathematical physics is the fractional diffusion equation; it generalizes the classical diffusion equation, treating super-diffusive flow processes; it becomes increasingly sought after in recent years. The old version of the diffusion process problem is of great application in many disciplines and aims to capture the historical status of the physical field from its current data. The most simple model of the time-diffusion problem is the well-known partial differential equation of the temperature field $u(x, t)$ governed by [6-8]

$$\frac{\partial u}{\partial u} = \frac{\partial^2 u}{\partial x^2}, \quad t \in \mathbb{R}^+, x \in \Omega \subset \mathbb{R}^1$$

under certain specified boundary conditions on $\partial\Omega$, and a given pre-specified initial condition at $t = 0$. The target is to find a closed form of $u(x, t)$ based only on the given initial and boundary conditions.

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Harold T. Davis created the notation for fractional calculus, which is a generalization of integration and differentiation to the noninteger-order fundamental operator ${}_a\mathbf{D}_t^{\alpha}$, where a and t are the operation's limits and $\alpha \in \mathbf{R}$. Fractional derivatives have been defined using various definitions, including Riemann-Liouville, Caputo, Hadamard, Erdélyi-Kober, Grünwald-Letnikov, Marchaud, and Riesz, to mention a few. The Riemann-Liouville, Caputo, and Grünwald-Letnikov definitions are the three best standard definitions for the universe of fractional calculus [9-11].

Due to the great demand for finding efficient solutions for time-fractional diffusion equations, recently, many researchers devoted their interest to handling this equation, for instance, the Regularization method [12], B-spline method [13], Finite difference method [14, 15], discontinuous Galerkin method [16, 17], collocation method [18, 19], fifth-kind Chebyshev tau method [20], Generalized Lucas tau method [21] and fifth-kind Chebyshev Galerkin method [22].

Let's stress that the "global dependence" problem has generally restricted the computation of the numerical solution of time-fractional differential equations to basic scenarios (low spatial dimension or short time integration). The solution at a time t_k typically depends on the solutions at all earlier time levels according to the concept of the fractional derivative. If low-order methods are used for spatial discretization, the storage would be expensive because all past solutions must be saved to compute the solution at the current time level. However, using the spectral approach [23-27], can relieve this storage restriction because it uses fewer grid points to provide highly accurate results than a low-order method. This reason motivates us to build and implement a robust and efficient Perov-Galerkin procedure that smoothly reduces the underlying time-fractional differential equation to a system of algebraic equations that can be efficiently inverted via Gaussian elimination procedures.

Lucas polynomials have vital places in the theory and practice of mathematics; there is a great deal of interest in their view and use in modern science. As a result, numerous academics have investigated their varied arithmetical features and produced several significant findings. For example, Abd-Elhameed and Napoli derived novel formulae on Lucas polynomials in [28], Abd-Elhameed et al. [29], derived novel formulas on Fibonacci and Lucas polynomials, Gumgum et al. [30] suggested Lucas polynomials collocation method for solving functional integro-differential equations, Youssri et al. executed a spectral Lucas approach for handling second-order boundary value problems [31], Youssri et al. [32] proposed a generalized Lucas Galerkin method for handling the linear one-dimensional telegraph type equation.

To the best of our knowledge, there are some advantages of the proposed technique that can be mentioned as follows:

• Selecting the basis functions in terms of the Lucas polynomials enables one to obtain approximate solutions with high accuracy by taking a few numbers of the retained modes. This leads to less computational time and computational errors.

• The proposed method has inverse factorial order.

The above advantages motivate our interest to employ the Lucas polynomials. In addition, the numerical investigations based on the Lucas polynomials are few. This also gives us another motivation to utilize numerically this kind of polynomials.

The paper is structured as follows: In Sect. 2, some necessary mathematical tools of fractional calculus and Lucas polynomials required for the construction of the work will be given. Sect. 3 is the main section, devoted to constructing a Pertov-Galerkin Procedure for treating the time-fractional diffusion equation. In Sect. 4, we study the convergence rate of the expansion coefficients and the truncation error as well. Some test problems are executed in Sect. 5, finally, some concluding remarks are reported in Sect. 6.

2. Some used definitions and formulas

This section is devoted to accounting for Caputo fractional derivative properties. Moreover, we give an important definitions and elementary relations concerning the family of the non orthogonal polynomials, namely, Lucas Polynomials (LPs).

2.1 *Some properties of the Caputo fractional derivative*

Definition 1 The fractional-order derivative in Caputo sense is defined as ([33]):

$$D^{\nu} g(x) = \frac{1}{\Gamma(r-\nu)} \int_0^x (x-t)^{r-\nu-1} g^{(r)}(t) dt, \ \ \nu > 0, \ x > 0, \tag{1}$$

$$r - 1 \leqslant \nu < r, \ r \in \mathbb{N}.$$

The last definition enables one to write

$$D^{\nu} C = 0, \ \ (C \text{ is a constant}), \tag{2}$$

$$D^{\nu} x^{k} = \begin{cases} 0, & \text{if } k \in \mathbb{N}_0 \text{ and } k < \lceil \nu \rceil, \\ \dfrac{\Gamma(k+1)}{\Gamma(k+1-\nu)} x^{k-\nu}, & \text{if } k \in \mathbb{N}_0 \text{ and } k \geq \lceil \nu \rceil, \end{cases} \tag{3}$$

where $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}_0 = \{0, 1, 2, ...\}$.

2.2 *An account on the LPs*

The recurrence relation of LPs $L_i(x)$ is defined as [34] as

$$L_i(x) = x L_{i-1}(x) + L_{i-2}(x), \ i \geq 2, \tag{4}$$

where

$$L_0(x) = 2 \text{ and } L_1(x) = x. \tag{5}$$

The analytic form of the $L_i(x)$ is

$$L_i(x) = i \sum_{r=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \frac{\binom{i-r}{r}}{i-r} x^{i-2r}, \ i \geq 1, \tag{6}$$

or, in another form

$$L_i(x) = 2i \sum_{k=0}^{i} \frac{\delta_{i+k} \binom{\frac{i+k}{2}}{\frac{i-k}{2}}}{i+k} x^{k}, \ i \geq 1, \tag{7}$$

where

$$\delta_r = \begin{cases} 1, & \text{if } r \text{ even}, \\ 0, & \text{if } r \text{ odd}. \end{cases} \tag{8}$$

The Binet's form for $L_i(x)$ can be written in the following form

$$L_i(x) = \frac{\left(x+\sqrt{x^2+4}\right)^i + \left(x-\sqrt{x^2+4}\right)^i}{2^i}, \ i \geq 0. \tag{9}$$

It is worth reporting that, the well-known Lucas numbers L_i can be obtained from LPs by setting $x = 1$, or can be generated from the Fibonacci recurrence relation $L_{i+2} = L_{i+1} + L_i$; $L_0 = 2$, $L_1 = 1$.

3. Petrov-Galerkin approach for the time-fractional diffusion equation

In this section, we consider the following Time-Fractional Diffusion Equation (TFDE) [13]

$$D_t^{\alpha} u(x,t) - \beta u_{xx}(x,t) = f(x,t), \ 0 < \alpha \leq 1, \tag{10}$$

subject to the Initial Condition (IC)

$$u(x,0) = \sigma(x), \ 0 < x \leq 1, \tag{11}$$

and the Homogeneous Boundary Conditions (HBCs)

$$u(0,t) = u(1,t) = 0, \ 0 < t \leq 1, \tag{12}$$

where β is arbitrary positive constant and $f(x, t)$ is the source term.

3.1 *Basis functions*

Consider the following basis functions

$$\lambda_i(x) = x L_{i+2} - L_{i+2}(x) + 2a_i(x), \tag{13}$$

where

$$a_i(x) = \begin{cases} 0, & \text{if } i \text{ odd}, \\ 1-x, & \text{otherwise}, \end{cases} \tag{14}$$

along with $L_i(t)$ defined in (6).

Corollary 1 $\lambda_i(x)$ can be written alternatively in the following form

$$\lambda_i(x) = -\sum_{k=0}^{\left\lfloor \frac{i}{2} \right\rfloor + 1} \frac{(i+2)\binom{i-k+2}{k}}{i-k+2} x^{i-2k+2} + \tau_i(x), \tag{15}$$

where

$$\tau_i(x)=\begin{cases} xL_{i+2}(1), & \text{if } i \text{ odd},\\ xL_{i+2}(1)-2x+2, & \text{otherwise},\end{cases} \tag{16}$$

and L_i is the so called Lucas numbers.

Corollary 2 For all $k \geq 0$, we have

$$\frac{d^2\lambda_i(x)}{dx^2}=-\sum_{k=0}^{\left\lfloor \frac{i}{2}\right\rfloor+1}\frac{(i+2)(i-2k+2)(i-2k+1)\binom{i-k+2}{k}}{i-k+2}x^{i-2k}. \tag{17}$$

3.2 *Petrov-Galerkin solution for TFDE*

Now, one may set

$$\begin{aligned} S_M &= \operatorname{span}\{\lambda_i(x)L_j(t): i, j = 0,1,\dots,M\},\\ V_M &= \{u\in S_M : u(0,t)=u(1,t)=0\}, \end{aligned} \tag{18}$$

then, any function $u(x, t) \in V_M$ may be written as

$$u_M(x,t)=\sum_{i=0}^{M}\sum_{j=0}^{M}c_{ij}\,\lambda_i(x)L_j(t). \tag{19}$$

Now, the application of Petrov-Galerkin is used to find $u_M(x, t) \in V_M$ such that

$$((D_t^{\alpha}u_M(x,t),x^r t^s))-\beta((u_{M\,xx}(x,t),x^r t^s))=((f(x,t),x^r t^s)),\ \ 0\le r\le M,\ 0\le s\le M-1, \tag{20}$$

where

$$((u(x,t),v(x,t)))=\int_0^1\int_0^1 u(x,t)v(x,t)\,dx\,dt, \tag{21}$$

$$(u(x),v(x))=\int_0^1 u(x)v(x)\,dx.$$

Now, for $0 \leq r \leq M$, $0 \leq s \leq M - 1$, Eq. (20) can be rewritten alternatively as

$$\sum_{i=0}^{M}\sum_{j=0}^{M}c_{ij}\left(\lambda_i(x),x^r\right)\left(D_t^{\alpha}L_j(t),t^s\right)-\beta\sum_{i=0}^{M}\sum_{j=0}^{M}c_{ij}\left(\lambda_i''(x),x^r\right)\left(L_j(t),t^s\right)=\left(\left(f(x,t),x^r t^s\right)\right). \tag{22}$$

Or, in a simple form

$$\sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij}\, g_{i,r}\, b_{j,s} - \beta \sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij}\, d_{i,r}\, h_{j,s} = f_{r,s}, \; 0 \le r \le M, \; 0 \le s \le M-1, \tag{23}$$

where

$$b_{j,s} = \left(D_t^{\alpha} L_j(t), t^s\right), \; d_{i,r} = \left(\lambda_i''(x), x^r\right), \; g_{i,r} = \left(\lambda_i(x), x^r\right), \; h_{j,s} = \left(L_j(t), t^s\right) \text{ and } f_{r,s} = \left(\left(f(x,t), x^r t^s\right)\right).$$

In addition, the IC (11) implies that

$$\sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij} \lambda_i \left(\frac{k+1}{M+2}\right) L_j(0) = \sigma\left(\frac{k+1}{M+2}\right), \; k: 0, \ldots, M. \tag{24}$$

And hence, Eqs. (23) and (24), produce a linear system of algebraic equations of dimension $(M + 1) \times (M + 1)$ in the unknown expansion coefficients c_{ij}, that may be solved using suitable procedure.

Theorem 1 The elements $b_{j,s}$, $d_{i,r}$, $g_{i,r}$ and $h_{j,s}$ are given by

$$b_{j,s} = \sum_{k=1}^{j} \frac{(2j)k! \begin{pmatrix} \frac{j+k}{2} \\ \frac{j-k}{2} \end{pmatrix} \delta_{j+k}}{(j+k)\Gamma(k-\alpha+1)(k-\alpha+s+1)},$$

$$d_{i,r} = -\frac{(i+1)(i+2)}{i+r+1}\; {}_3F_2\left(\left.\begin{matrix} \frac{1}{2}-\frac{i}{2}, -\frac{i}{2}, -\frac{i}{2}-\frac{r}{2}-\frac{1}{2} \\ -i-1, -\frac{i}{2}-\frac{j}{2}+\frac{1}{2} \end{matrix}\right| -4\right),$$

$$g_{i,r} = -\frac{1}{i+r+3}\; {}_3F_2\left(\left.\begin{matrix} -\frac{i}{2}-1, -\frac{i}{2}-\frac{1}{2}, -\frac{i}{2}-\frac{r}{2}-\frac{3}{2} \\ -i-1, -\frac{i}{2}-\frac{r}{2}-\frac{1}{2} \end{matrix}\right| -4\right) + \varsigma_{i,r}, \tag{25}$$

$$h_{j,s} = \frac{1}{j+s+1}\; {}_3F_2\left(\left.\begin{matrix} \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -\frac{j}{2}-\frac{s}{2}-\frac{1}{2} \\ 1-j, -\frac{j}{2}-\frac{s}{2}+\frac{1}{2} \end{matrix}\right| -4\right),$$

where

$$\varsigma_{i,r} = \begin{cases} \dfrac{L_{i+2}}{r+2}, & \text{if } i \text{ odd}, \\ \dfrac{2+(r+1)L_{i+2}}{(r+1)(r+2)}, & \text{otherwise}, \end{cases} \tag{26}$$

and ${}_rF_s$ denotes the Gauss generalized hypergeometric function defined by

$${}_rF_s\left(\begin{matrix} p_1, p_2, \cdots, p_r \\ q_1, q_2, \cdots, q_s \end{matrix}\middle| t\right) = \sum_{n=0}^{\infty} \frac{(p_1)_n (p_2)_n \cdots (p_r)_n}{(q_1)_n (q_2)_n \cdots (q_s)_n} \frac{t^n}{n!}. \tag{27}$$

Proof. To find the elements of $b_{j,s}$, by using Definition 1, we get

$$b_{j,s} = \left(D_t^{\alpha} L_j(t), t^s\right)$$

$$= \sum_{k=1}^{j} \frac{(2j)k! \begin{pmatrix} \frac{j+k}{2} \\ \frac{j-k}{2} \end{pmatrix} \delta_{j+k}}{(j+k)\Gamma(k-\alpha+1)} \int_0^1 t^{k-\alpha+s} dt$$

$$= \sum_{k=1}^{j} \frac{(2j)k! \begin{pmatrix} \frac{j+k}{2} \\ \frac{j-k}{2} \end{pmatrix} \delta_{j+k}}{(j+k)\Gamma(k-\alpha+1)(k-\alpha+s+1)}. \tag{28}$$

Now, to find the elements of $d_{i,r}$, by using Corollary 2, one has

$$d_{i,r} = \left(\lambda_i''(x), x^r\right)$$

$$= -\sum_{k=0}^{\left\lfloor \frac{i}{2} \right\rfloor + 1} \frac{(i+2)(i-2k+2)(i-2k+1)\binom{i-k+2}{k}}{i-k+2} \int_0^1 x^{i+r-2k} dx$$

$$= -\sum_{k=0}^{\left\lfloor \frac{i}{2} \right\rfloor + 1} \frac{(i+2)(i-2k+2)(i-2k+1)\binom{i-k+2}{k}}{(i-k+2)(i+r-2k+1)}, \tag{29}$$

after simplifying the right-hand side of the last equation, we get the desired result. Finally, we can find similarly the elements of $g_{i,r}$ and $h_{j,s}$, after using Corollary 1 and the analytic form (7). □

Remark 1 The following transformation

$$v(x,t) = u(x,t) - (1-x)u(0,t) - xu(1,t), \tag{30}$$

helps us transform the boundary conditions from non-homogeneous ones to homogeneous ones.

Remark 2 Algorithm 1 shows all the steps required to obtain the numerical solution of Eq. (10) governed by the conditions (11)-(12).

Algorithm 1 Coding algorithm for the proposed technique
Input α, β, $\sigma(x)$ and $f(x, t)$.
Step 1. Assume an approximate solution $u_M(x, t)$ as in (19).
Step 2. Apply the Petrov-Galerkin method to obtain the system in (23) and (24).
Step 3. Use Theorem 1 to get the elements $b_{j,s}$, $d_{i,r}$, $g_{i,r}$ and $h_{j,s}$.
Step 4. Use *NDsolve* command to solve the system in (23) and (24) to get c_{ij}.
Output $u_M(x, t)$

Remark 3 We would like to report here that our proposed method can be used to solve the two-dimensional time diffusion equation

$$u_t(x,y,t) - u_{xx}(x,y,t) - u_{yy}(x,y,t) = K(x,y,t), \tag{31}$$

subject to the IC

$$u(x,y,0) = h(x,y), \ 0 < x, y \le 1, \tag{32}$$

and the HBCs

$$\begin{aligned} u(0,y,t) &= u(1,y,t) = 0, \ 0 < y, t \le 1, \\ u(x,0,t) &= u(x,1,t) = 0, \ 0 < x, t \le 1, \end{aligned} \tag{33}$$

where $K(x, y, t)$ is the source term.

In this case, we assume $u_M(x,y,t) = \sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{M} a_{ijk}\,\lambda_i(x)\lambda_j(y)L_k(t)$, and imitating similar steps as in Section 3 to get a linear system of algebraic equations of dimension $(M + 1)^3$ in the unknown expansion coefficients a_{ijk}, which can be inverted via Gaussian elimination procedure.

4. Convergence rate and truncation error bound

In this section, we find an upper estimate for the expansion coefficients c_{ij}, to specify the convergence rate, and then we find a dominant bound for the truncation error, for the previous purposes, we need the following results.

Lemma 1 For all $t \in [0, 1]$, the following inequality is valid:

$$L_j(t) \le \frac{1}{2}\phi^j, \ \forall j \ge 0,$$

where ϕ is the well-known golden ratio and $\phi = \frac{1+\sqrt{5}}{2}$.

Proof. Noting that $L_j(z) \leq L_j(1) = L_j$, and since $\lim_{n\to\infty} \frac{L_n}{\phi^n} = \frac{1}{2}$, we get the desired result. □

Lemma 2 For all $x \in [0, 1]$, we have the following inequality

$$\lambda_i(x) \leq \phi^{2i}, \ \forall i \geq 0$$

Proof. By mathematical induction on i, we have $\lambda_i(x) \leq \phi^i x(1-x)L_i(x)$, now noting that $x(1-x) \leq \frac{1}{4}$ and direct application of Lemma 1, we get the desired result. □

The following theorem [34] is needed.

Theorem 2 If $f(t)$ is defined on [0, 1] and $|f^{(j)}(0)| \leq \eta^j$, $j \geq 0$, where η is a positive constant, and if $f(t)$ has the expansion $f(t) = \sum_{j=0}^{\infty} a_j L_j(t)$, then one has:

$$|a_j| \leq \frac{\eta^j \cosh(2\eta)}{j!}.$$

Now, we are ready to prove the following theorem.

Theorem 3 If $v(x) = x(1-x)f(x)$ is defined over [0, 1] satisfy the homogeneous boundary conditions $v(0) = v(1) = 0$, $f(x)$ satisfy the hypotheses of Theorem 2, and if $v(x)$ has the expansion $v(x) = \sum_{i=0}^{\infty} b_i \lambda_i(x)$, then one has:

$$|b_i| \leq \frac{\phi^{-i} \eta^i \cosh(2\eta)}{i!}.$$

Proof. With the aid of hypotheses of Theorem 3 and the following inequality

$$\lambda_i(x) \leq \phi^i x(1-x)L_i(x),$$

we can write

$$v(x) = x(1-x)f(x) = \sum_{i=0}^{\infty} b_i \lambda_i(x)$$

$$\leq \sum_{i=0}^{\infty} b_i \phi^i L_i(x). \tag{34}$$

Now, theorem 2 enables us to get the following inequality

$$|b_i \phi^i| \leq \frac{\eta^i \cosh(2\eta)}{i!},$$

which can be rewritten as

$$|b_i| \leq \frac{\phi^{-i} \eta^i \cosh(2\eta)}{i!}.$$

This completes the proof of this theorem. □

Theorem 4 If $u(x,t)=x(1-x)f_1(x)f_2(t)=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}c_{ij}\lambda_i(x)L_j(t)$ is defined over [0, 1] × [0, 1] with $|f_k^{(i)}(0)|\leq\zeta_k^i$, $k=1,2$, $i\geq 0$, ζ_k are positive constants, then the following inequality holds

$$|c_{ij}|\leq\frac{\phi^{-i}\zeta_1^i\zeta_2^j\cosh(2\zeta_1)\cosh(2\zeta_2)}{i!j!}$$

Proof. The proof of this theorem can be obtained by imitating similar steps as followed in Theorem 3 in [32]. □

Theorem 5 If $u(x, t)$ satisfies the hypothesis of Theorem 4, then we have the following upper estimate on the truncation error:

$$|u-u_M|\leq\frac{\cosh(2\zeta_1)\cosh(2\zeta_2)e^{\phi(\zeta_1+\zeta_2)}\left[(\phi\zeta_1)^{M+1}+(\phi\zeta_2)^{M+1}\right]}{2(M+1)!}.$$

Proof. From definitions of $u(x, t)$ and $u_M(x, t)$, we get

$$|u-u_M|=\left|\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}\lambda_i(x)L_j(t)-\sum_{i=0}^{M}\sum_{j=0}^{M}c_{ij}\lambda_i(x)L_j(t)\right|$$
$$\leq\left|\sum_{i=0}^{M}\sum_{j=M+1}^{\infty}c_{ij}\lambda_i(x)L_j(t)\right|+\left|\sum_{i=M+1}^{\infty}\sum_{j=0}^{\infty}c_{ij}\lambda_i(x)L_j(t)\right|. \tag{35}$$

By virtue of Theorem 4 along with two Lemmas 1 and 2, one has

$$|u-u_M|\leq\frac{\cosh(2\zeta_1)\cosh(2\zeta_2)}{2}\left[\sum_{i=0}^{M}\frac{(\phi\zeta_1)^i}{i!}\sum_{j=M+1}^{\infty}\frac{(\phi\zeta_2)^j}{j!}+\sum_{i=M+1}^{\infty}\frac{(\phi\zeta_1)^i}{i!}\sum_{j=0}^{\infty}\frac{(\phi\zeta_2)^j}{j!}\right]. \tag{36}$$

With the aid of the following inequalities

$$\sum_{i=0}^{M}\frac{(\phi\zeta_1)^i}{i!}=\frac{e^{\phi\zeta_1}\Gamma(M+1,\phi\zeta_1)}{M!}<e^{\phi\zeta_1},$$
$$\sum_{j=M+1}^{\infty}\frac{(\phi\zeta_2)^j}{j!}=e^{\phi\zeta_2}\left(1-\frac{\Gamma(M+1,\phi\zeta_2)}{M!}\right)<\frac{e^{\phi\zeta_2}(\phi\zeta_2)^{M+1}}{(M+1)!},$$
$$\sum_{i=M+1}^{\infty}\frac{(\phi\zeta_1)^i}{i!}=e^{\phi\zeta_1}\left(1-\frac{\Gamma(M+1,\phi\zeta_1)}{M!}\right)<\frac{e^{\phi\zeta_1}(\phi\zeta_1)^{M+1}}{(M+1)!},$$
$$\sum_{j=0}^{\infty}\frac{(\phi\zeta_2)^j}{j!}=e^{\phi\zeta_2}, \tag{37}$$

we get the following estimation

$$|u-u_M| \leq \frac{\cosh(2\zeta_1)\cosh(2\zeta_2)e^{\phi(\zeta_1+\zeta_2)}\left[(\phi\zeta_1)^{M+1}+(\phi\zeta_2)^{M+1}\right]}{2(M+1)!}, \tag{38}$$

where Γ(.) and Γ(., .) denote, respectively, gamma and upper incomplete gamma functions [35]. This completes the proof of this theorem. □

5. Illustrative examples

Test Problem 1 [13, 36, 37] Consider the TFDE of the form

$$D_t^{\alpha}u(x,t)-u_{xx}(x,t)+f(x,t),\ 0<\alpha\leq 1, \tag{39}$$

subject to the IC

$$u(x,0)=0,\ 0<x\leq 1, \tag{40}$$

and the HBCs

$$u(0,t)=u(1,t)=0,\ 0<t\leq 1, \tag{41}$$

where $f(x,t)=\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}\sin(2\pi x)+4\pi^2t^2\sin(2\pi x)$ and the exact solution of this problem is $u(x, t) = t^2\sin(2\pi x)$. In Table 1, we give a comparison of Maximum Absolute Errors (MAE) between our method and the method in [36] at $\alpha = 0.5$. Also, we give a comparison of L_2 error between our method and method in [37] at $\alpha = 0.5$ in Table 2. Figure 1 shows the exact and approximate solutions at $M = 7$ and $\alpha = 0.2$. Table 3, presents the Absolute Error (AE) at different values of (x, t) when $M = 7$ and $\alpha = 0.7$. Table 4 shows the AE at different values of t when $M = 7$ and $\alpha = 0.8$. Finally, Figure 2 illustrate the L_∞ error at different values of α at $M = 7$.

Table 1. Comparison of the MAE for Example 1

Method in [36] at $\Delta t = 0.001$		Our method at $M = 7$
M	MAE	MAE
4	1.72×10^{-1}	
8	4.80×10^{-2}	
16	1.23×10^{-2}	3.18816×10^{-5}
32	3.08×10^{-3}	
64	7.70×10^{-4}	

Table 2. Comparison of the L_2 error for Example 1

Method in [37] at $\Delta t = 0.001$		Our method at $M = 7$
M	MAE	MAE
4	1.09×10^{-1}	
8	3.20×10^{-2}	
16	8.42×10^{-2}	1.00451×10^{-5}
32	2.15×10^{-3}	
64	5.42×10^{-4}	

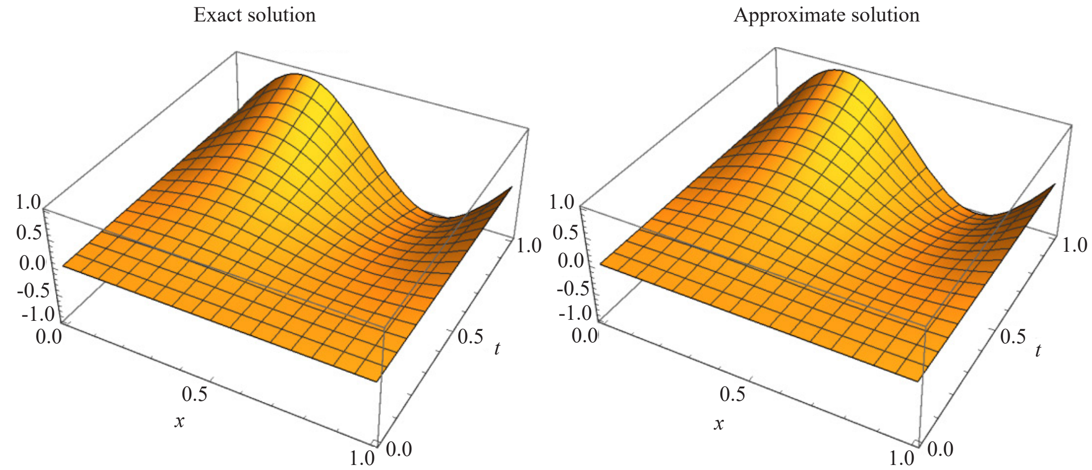

Figure 1. The exact and the approximate solutions for Example 1

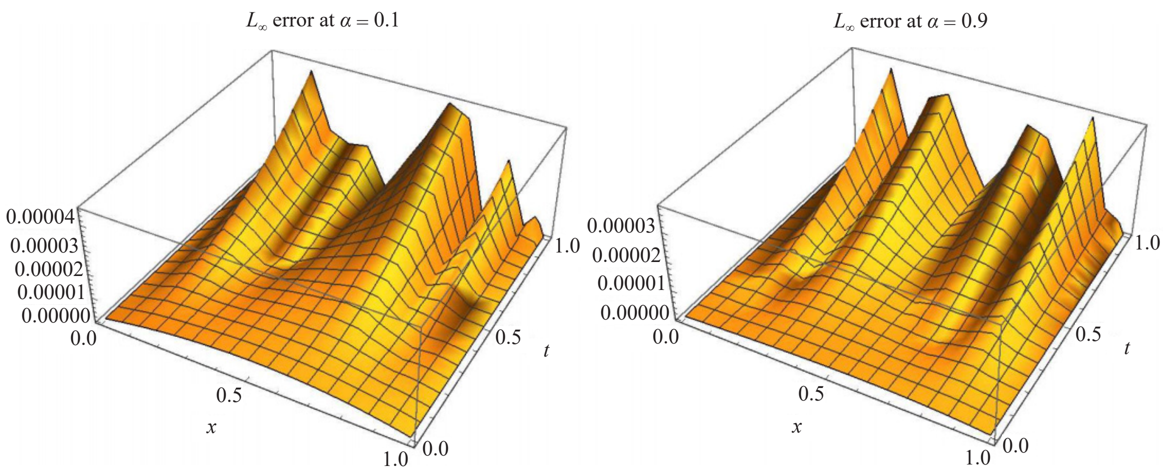

Figure 2. The L_∞ error for Example 1 at different values of α

Table 3. The AE of Example 1

(x, t)	(0.1, 0.1)	(0.2, 0.2)	(0.3, 0.3)	(0.4, 0.4)	(0.5, 0.5)	(0.6, 0.6)	(0.7, 0.7)	(0.8, 0.8)	(0.9, 0.9)
AE	1.73×10^{-6}	3.00×10^{-6}	8.27×10^{-6}	1.03×10^{-5}	8.18×10^{-6}	1.37×10^{-6}	6.42×10^{-6}	2.02×10^{-6}	2.45×10^{-5}

Table 4. The AE of Example 1 at $M = 7$

	$\alpha = 0.8$				
x	$t = \frac{1}{10}$	$t = \frac{3}{10}$	$t = \frac{5}{10}$	$t = \frac{7}{10}$	$t = \frac{9}{10}$
0.1	1.97844×10^{-7}	2.01871×10^{-6}	5.69922×10^{-6}	1.12279×10^{-5}	1.85852×10^{-5}
0.2	6.45779×10^{-8}	1.4729×10^{-6}	4.32913×10^{-6}	8.61085×10^{-6}	1.42823×10^{-5}
0.3	5.55413×10^{-7}	3.19976×10^{-6}	8.47339×10^{-6}	1.64081×10^{-5}	2.70516×10^{-5}
0.4	5.88208×10^{-7}	2.40135×10^{-6}	6.0049×10^{-6}	1.14347×10^{-5}	1.87439×10^{-5}
0.5	5.19608×10^{-7}	6.72478×10^{-7}	9.03347×10^{-7}	1.24591×10^{-6}	1.75241×10^{-6}
0.6	4.12898×10^{-7}	1.10716×10^{-6}	4.26828×10^{-6}	9.04185×10^{-6}	1.53813×10^{-5}
0.7	3.29718×10^{-7}	2.0593×10^{-6}	6.94805×10^{-6}	1.43126×10^{-5}	2.41144×10^{-5}
0.8	7.50816×10^{-7}	2.35222×10^{-6}	5.4988×10^{-6}	1.02097×10^{-5}	1.65135×10^{-5}
0.9	5.93441×10^{-7}	2.52173×10^{-6}	6.36324×10^{-6}	1.21292×10^{-5}	1.98348×10^{-5}

Test Problem 2 [13, 37] Consider the TFDE of the form

$$D_t^{\alpha} u(x,t) - u_{xx}(x,t) + f(x,t), \ 0 < \alpha \le 1, \tag{42}$$

subject to the IC

$$u(x,0) = 0, \ 0 < x \le 1, \tag{43}$$

and the HBCs

$$u(0,t) = u(1,t) = 0, \ 0 < t \le 1, \tag{44}$$

where $f(x, t)$ is chosen such that the exact solution of this problem is $u(x, t) = \sin(\pi x)\sin(\pi t)$.

In Table 5, we give a comparison of MAE between our method and method in [13] at $\alpha = 0.5$. Also, Figure 3 shows the approximate and exact solutions at $\alpha = 0.5$ and $M = 8$. In Figure 4, we plot the L_∞ errors at different values of α when $M = 8$. Table 6 presents the AE at $\alpha = 0.6$ and $M = 8$ for different values of t. Figure 5 illustrates the approximate solution (Left) and L_∞ error (Right) at $\alpha = 0.9$ and $M = 8$. At the end, the AE at different values of (x, t) at $M = 7$ and $M = 8$ is presented in Table 7 at $\alpha = 0.1$.

Table 5. Comparison of the MAE for Example 2

α	Method in [13] at $M = 32$ and $\Delta x = 0.001$	Our method at $M = 8$
0.2	1.42×10^{-4}	6.03935×10^{-6}
0.5	1.10×10^{-3}	1.15864×10^{-6}
0.7	3.21×10^{-3}	2.60955×10^{-6}

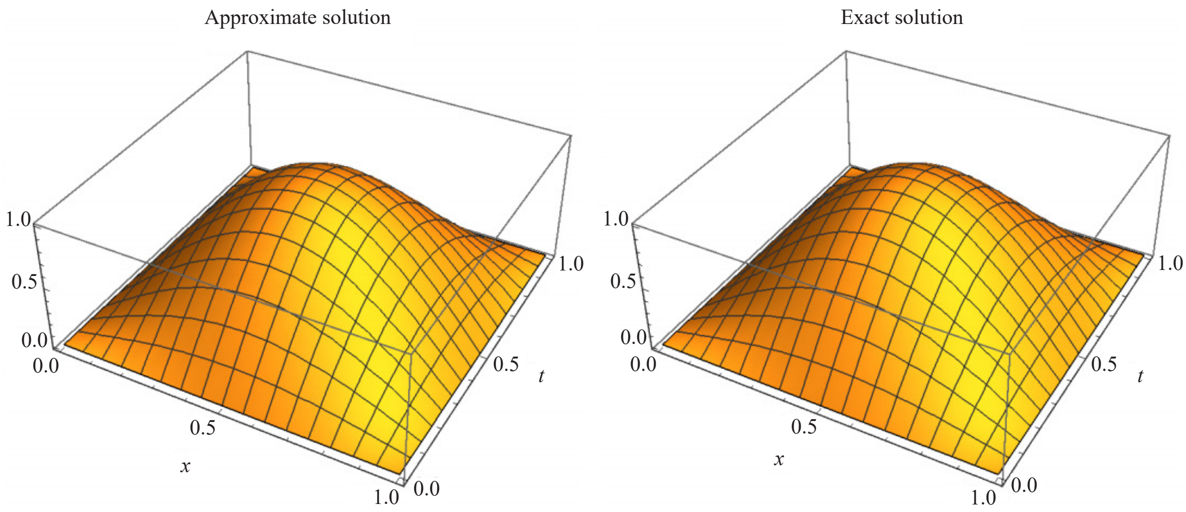

Figure 3. The approximate and exact solutions for Example 2

Table 6. The AE of Example 2

x	$\alpha = 0.6$			Convergence rate
	$t = \frac{3}{10}$	$t = \frac{6}{10}$	$t = \frac{9}{10}$	
0.1	5.83998×10^{-7}	8.42798×10^{-7}	1.26045×10^{-6}	10^{-6}
0.2	1.09395×10^{-6}	1.5717×10^{-6}	2.34284×10^{-6}	10^{-6}
0.3	1.51245×10^{-6}	2.16457×10^{-6}	3.21756×10^{-6}	10^{-6}
0.4	1.82186×10^{-6}	6.07153×10^{-6}	3.85024×10^{-6}	10^{-6}
0.5	2.0043×10^{-6}	2.59797×10^{-6}	4.20186×10^{-6}	10^{-6}
0.6	2.03514×10^{-6}	2.84725×10^{-6}	4.22667×10^{-6}	10^{-6}
0.7	1.88689×10^{-6}	2.65699×10^{-6}	3.87442×10^{-6}	10^{-6}
0.8	1.52417×10^{-6}	2.13489×10^{-6}	3.08983×10^{-6}	10^{-6}
0.9	9.09067×10^{-7}	1.26558×10^{-6}	1.81662×10^{-6}	10^{-6}

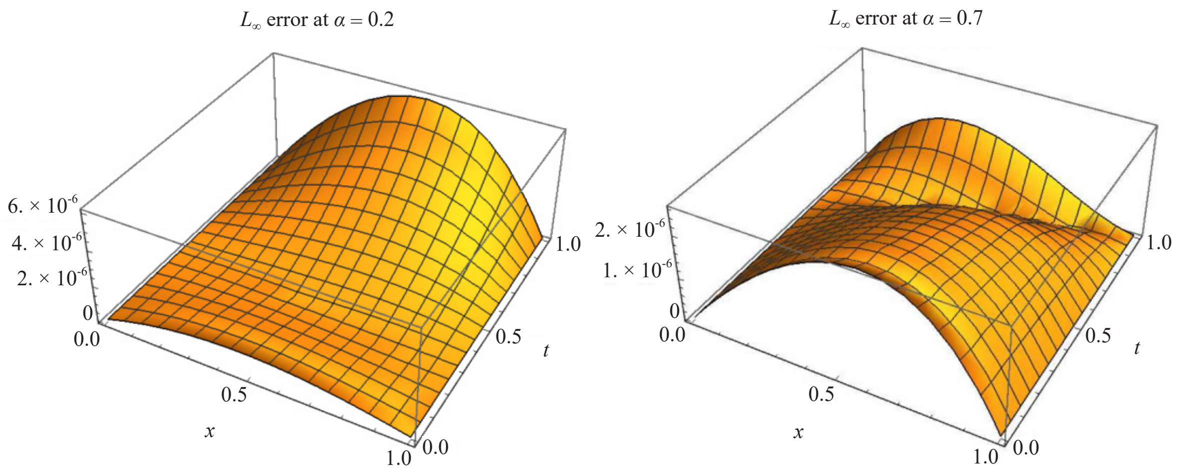

Figure 4. The L_∞ error for Example 2

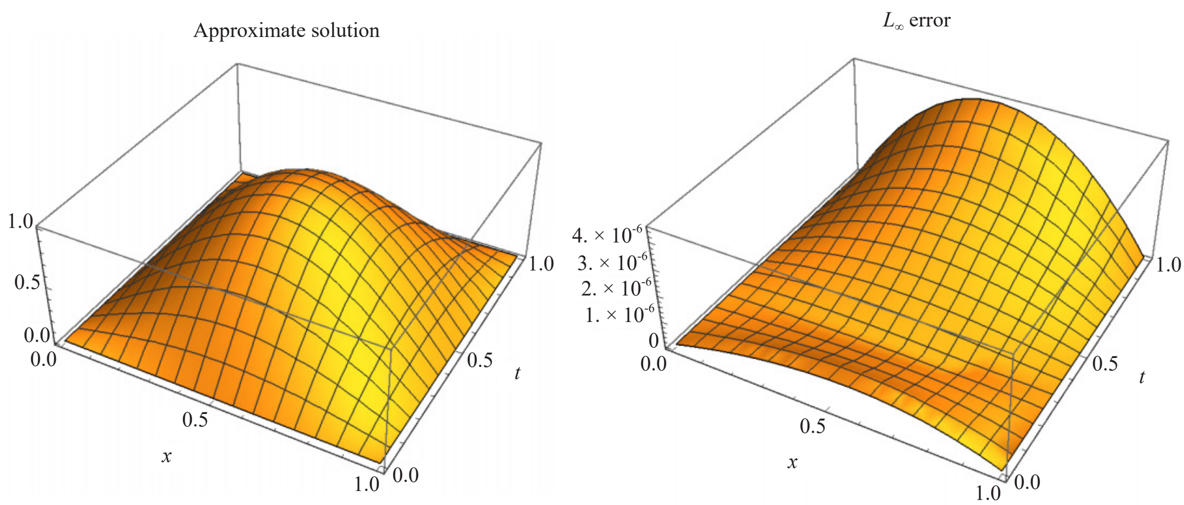

Figure 5. The approximate solution (Left) and L_∞ error (Right) for Example 2

Test Problem 3 [36] Consider the TFDE of the form

$$D_t^\alpha u(x,t) - u_{xx}(x,t) + f(x,t), \;\; 0 < \alpha \leq 1, \tag{45}$$

subject to the IC

$$u(x,0) = 0, \;\; 0 < x \leq 1, \tag{46}$$

and the HBCs

$$u(0,t) = u(1,t) = 0, \;\; 0 < t \leq 1, \tag{47}$$

where $f(x,t)=\frac{1}{\Gamma(2-\alpha)}t^{1-\alpha}x^2(1-x)+2t(3x-1)$ and the exact solution of this problem is $u(x, t) = tx^2(1 - x)$. The application of our method at $\alpha = 0.5$ for $M = 1$ yields the following system of equations

$$
\begin{aligned}
&c_{00}+0.459354c_{01}+2.56066c_{10}+1.15321c_{11}=0.69386,\\
&c_{00}+0.459354c_{01}+0.43934c_{10}+0.224849c_{11}=-0.234506,\\
&4c_{00}+\left(\sqrt{2}+6\right)c_{10}=0,\\
&4c_{00}=\left(\sqrt{2}-6\right)c_{10},
\end{aligned}
\tag{48}
$$

that yield $\{c_{00} = 0, c_{10} = 0, c_{01} = -1, c_{11} = 1\}$, and therefore $u_M(x, t) = tx^2(1 - x)$, which is the exact solution.

Table 7. The AE of Example 2

(x, t)	$\alpha = 0.1$		Convergence rate
	$M = 7$	$M = 8$	
(0.1, 0.1)	8.80217×10^{-7}	1.94195×10^{-8}	10^{-7}
(0.2, 0.2)	1.72101×10^{-6}	9.04169×10^{-8}	10^{-6}
(0.3, 0.3)	5.95336×10^{-6}	2.85192×10^{-7}	10^{-6}
(0.4, 0.4)	5.44695×10^{-6}	5.78739×10^{-7}	10^{-6}
(0.5, 0.5)	7.05598×10^{-6}	8.42112×10^{-7}	10^{-6}
(0.6, 0.6)	2.58616×10^{-7}	1.11252×10^{-6}	10^{-6}
(0.7, 0.7)	5.76057×10^{-6}	1.23026×10^{-6}	10^{-6}
(0.8, 0.8)	1.57319×10^{-6}	1.18253×10^{-6}	10^{-6}
(0.9, 0.9)	9.19882×10^{-7}	8.48789×10^{-7}	10^{-7}

Test Problem 4 Consider the two dimensional time diffusion equation

$$u_t(x,y,t)-u_{xx}(x,y,t)-u_{yy}(x,y,t)=K(x,y,t), \tag{49}$$

subject to the IC

$$u(x,y,0)=(x^2-x^3)(y^2-y^3),\ \ 0<x,y\le 1, \tag{50}$$

and the HBCs

$$u(0,y,t)=u(1,y,t)=0,\ \ 0<y,t\le 1,$$

$$u(x,0,t)=u(x,0,t)=0,\ \ 0<x,t\le 1, \tag{51}$$

where

$$K(x,y,t)=x^3\left(t(2-6y)+(y-3)y(y+2)+2\right)+x^2\left(t(6y-2)+y(-y^2+y+6)-2\right)$$
$$-6(t+1)x(y-1)y^2+2(t+1)(y-1)y^2, \tag{52}$$

and the exact solution of this problem is $u(x,y,t)=(x^2-x^3)(y^2-y^3)(1+t)$.

The application of our method at $M=1$ yields the following system of equations

$$5\left(240a_{000}+65a_{001}+3(544a_{010}+147a_{011}+544a_{100}+147a_{101}+3696a_{110})\right)+14949a_{111}=925,$$
$$960a_{000}+255a_{001}+4440a_{010}+1195a_{011}+6480a_{100}+1719a_{101}+30120a_{110}+8091a_{111}=735,$$
$$960a_{000}+255a_{001}+6480a_{010}+1719a_{011}+4440a_{100}+1195a_{101}+30120a_{110}+8091a_{111}=735,$$
$$720a_{000}+189a_{001}+3480a_{010}+921a_{011}+3480a_{100}+921a_{101}+16320a_{110}+4369a_{111}=549, \tag{53}$$
$$2(9a_{000}+84a_{010}+a_{100})+784a_{110}=9,$$
$$18a_{000}+93a_{010}+168a_{100}+868a_{110}=9,$$
$$18a_{000}+168a_{010}+93a_{100}+868a_{110}=9,$$
$$36a_{000}+186a_{010}+a_{100}+961a_{110}=18,$$

that yield $\{a_{000}=\frac{1}{2}, a_{001}=1, a_{010}=0, a_{011}=0, a_{100}=0, a_{101}=0, a_{110}=0, a_{111}=0\}$, and therefore $u_M(x,y,t)=(t+1)\,x^2(1-x)\,y^2(1-y)$, which is the exact solution.

6. Ending concluding remarks

We summarize the main findings of this work as follows; we have suggested, implemented, and analyzed an accurate spectral Perov-Galerkin Lucas polynomials scheme for efficiently solving the time-fractional diffusion equation, the error analysis is established, numerical test problems verify the findings of the work, we aim in near future to use the offered algorithm to handle more complicated models in the science of partial differential equations, for instance, [38-40].

Conflict of interest

The authors declare no competing financial interest.

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