



## Research Article

# Enriched Pro-Categories and Shapes

Nikica Uglešić 

23287 Veli RAT, Sv. Ante 9, Dugi Otok, Croatia  
E-mail: [uglesic@pmfst.hr](mailto:uglesic@pmfst.hr)

**Received:** 14 January 2022; **Revised:** 17 March 2022; **Accepted:** 06 April 2022

**Abstract:** Given a category  $\mathcal{C}$  and a directed partially ordered set  $J$ , a certain category  $pro^J\text{-}\mathcal{C}$  on inverse systems in  $\mathcal{C}$  is constructed such that the ordinary pro-category  $pro\text{-}\mathcal{C}$  is the most special case of a singleton  $J \equiv \{1\}$ . Further, the known pro\*-category  $pro^*\text{-}\mathcal{C}$  becomes  $pro^{\mathbb{N}}\text{-}\mathcal{C}$ . Moreover, given a pro-reflective category pair  $(\mathcal{C}, \mathcal{D})$ , the  $J$ -shape category  $Sh_{(\mathcal{C}, \mathcal{D})}^J$  and the corresponding  $J$ -shape functor  $S^J$  are constructed which, in mentioned special cases, become the well known ones. Among several important properties, the continuity theorem for a  $J$ -shape category is established. It implies the “ $J$ -shape theory” is a genuine one such that the shape and the coarse shape theory are its very special instances.

**Keywords:** partially ordered set, category, functor, pro-category, pro-reflective subcategory, (abstract) shape, (abstract) coarse shape

**MSC:** 55P55, 18A32

## 1. Introduction

The shape theory, from the very beginning, has been an operable extension and generalization of the homotopy theory to the class of all (locally bad) topological spaces. Since Borsuk’s paper [1] and book [2], many articles ([3-10] are some of the most fundamental) and several books [11-14] concerning shape theory were written almost in the first decade already. By attempting to describe the shape theory (standard and abstract) as an axiomatic homotopy theory (founded by D. G. Quillen [15]), the strong shape theory has been obtained [16-18]. At the same time some shape theorists introduced and considered several classification of metrizable compacta coarser than the shape type. The most interesting of them are the Borsuk’s quasi-equivalence [19] and Mardešić  $S$ -equivalence [20]. They were further studied by the author and some others ([21-23]). On that line, the most important has become a certain uniformization of the  $S$ -equivalence, called the  $S^*$ -equivalence, which admits a categorical characterization, [23]. Moreover, it admits (genuine and different; [24-25]) generalizations to all topological spaces as well as to any abstract categorical framework [29], and all the well known shape invariants remain as the invariants of the both generalizations (in addition, [27]).

In this paper we generalize the generalization introduced in [19], the coarse shape theory, so that it and the shape theory as well become the very special cases of the new, so called,  $J$ -shape theory.

A part of the idea came from the recently founded quotient shape theory for a concrete category [28]. Namely, figuratively speaking, the quotient shapes of an object are “its (changeable) pictures” depending on the distance

of the “view point” which is determined by a “reciprocal” infinite cardinality (larger cardinal-closer distance, i.e., finer picture, and comparing them to the objects of lower cardinalities). This role hereby overtakes a directed partially ordered set  $J$  (larger set  $J$ -larger distance, i.e., coarser picture, and the comparing objects are those of  $\mathcal{D}$ ). In order to realize this idea, we have followed the construction of the coarse shape category obtained in [26]. Given a category  $\mathcal{C}$  and a directed partially ordered set  $J$ , in the First step (Section 3), each morphism set  $(inv\text{-}\mathcal{C})(X, Y)$  is essentially enriched, according to  $J$ , to the set  $(inv^J\text{-}\mathcal{C})(X, Y)$  making a new category  $inv^J\text{-}\mathcal{C}$  (with the same object class-all inverse systems in  $\mathcal{C}$ ). In the second step, on each set  $(inv^J\text{-}\mathcal{C})(X, Y)$  an equivalence relation is defined, according to  $J$ , that is compatible with the composition so that there is the corresponding quotient category  $(inv^J\text{-}\mathcal{C})/\sim$ , denoted by  $pro^J\text{-}\mathcal{C}$ . In the trivial case  $J = \{1\}$ ,  $pro^{\{1\}}\text{-}\mathcal{C} = pro\text{-}\mathcal{C}$ , while in the case of  $J = \mathbb{N}$ ,  $pro^{\mathbb{N}}\text{-}\mathcal{C} = pro^*\text{-}\mathcal{C}$  (of [26]). Then, for a suitable pair  $X, Y$  and an enough large  $J$ , in the set  $(pro^J\text{-}\mathcal{C})(X, Y)$  may exist an isomorphism, while there is no isomorphism in the set  $(pro\text{-}\mathcal{C})(X, Y)$ . Finally, in the third step (Section 4), given a pro-reflective subcategory pair  $\mathcal{D} \subseteq \mathcal{C}$ , the construction of the appropriate  $J$ -shape category  $Sh_{(\mathcal{C}, \mathcal{D})}^J$  and the  $J$ -shape functor  $S^J : \mathcal{C} \rightarrow Sh_{(\mathcal{C}, \mathcal{D})}^J$  follows by the usual standard pattern. Clearly, in the mentioned special case,  $Sh_{(\mathcal{C}, \mathcal{D})}^{\{1\}} = Sh_{(\mathcal{C}, \mathcal{D})}$  (the abstract shape category of [13]) and  $Sh_{(\mathcal{C}, \mathcal{D})}^{\mathbb{N}} = Sh_{(\mathcal{C}, \mathcal{D})}^*$  (the abstract coarse shape category of [29]) having their realizing categories  $pro^{\{1\}}\text{-}\mathcal{D} = pro\text{-}\mathcal{D}$  and  $pro^{\mathbb{N}}\text{-}\mathcal{D} = pro^*\text{-}\mathcal{D}$ .

In Section 5 we have proven the continuity theorem for every  $J$ -shape category. It strongly confirms that the  $J$ -shape theory is a genuine shape theory. At the end (Section 6) we have proven the full analogue of the well known Morita lemma of [9] that characterizes an isomorphism of  $pro^J\text{-}\mathcal{C}$ , which is then very useful for characterizing a  $J$ -shape isomorphism in the corresponding realizing category  $pro^J\text{-}\mathcal{D}$ .

Of course, the whole of this should be firstly applied to the pro-reflective category pair  $(HTop, HPol)$  and to its subpair  $(HcM, HcPol)$  (where only sequential expansions are needed).

## 2. Preliminaries

We assume that the notion of a pro-category is well known as well as the basics of the (abstract) shape theory, especially, via the inverse systems approach due to Mardešić and Segal [13]. For the sake of completeness, we shall briefly recall the needed notions and main facts concerning a  $pro^*$ -category and the coarse shape obtained in [26]. The category language follows [29].

Let  $\mathcal{C}$  be a category, and let  $inv\text{-}\mathcal{C}$  be the corresponding  $inv$ -category. Given a pair  $X, Y$  of inverse systems in  $\mathcal{C}$ , a  $*$ -morphism (originally, an  $S^*$ -morphism) of  $X$  to  $Y$ , denoted by

$$(f, f_\mu^n) : X = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \rightarrow (Y_\mu, q_{\mu\mu'}, M) = Y,$$

is an ordered pair consisting of a function  $f : M \rightarrow \Lambda$  (the index function) and, for each  $\mu \in M$ , of a sequence  $(f_\mu^n)$  of  $\mathcal{C}$ -morphisms  $f_\mu^n : X_{f(\mu)} \rightarrow Y_\mu$ ,  $n \in \mathbb{N}$ , satisfying the following condition:

$$(\forall \mu \leq \mu' \text{ in } M) (\exists \lambda \in \Lambda, \lambda \geq f(\mu), f(\mu')) (\exists n \in \mathbb{N}) (\forall n' \geq n)$$

$$f_\mu^{n'} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{n'} p_{f(\mu')\lambda}.$$

Clearly, the equality then holds for every  $\lambda' \geq \lambda$  as well. If the index function  $f$  is increasing and, for every pair  $\mu \leq \mu'$ , one may put  $\lambda = f(\mu')$  then  $(f, f_\mu^n)$  is said to be a simple  $*$ -morphism. If, in addition,  $M = \Lambda$  and  $f = 1_\Lambda$ , then  $(1_\Lambda, f_\mu^n)$  is said to be a level  $*$ -morphism. Finally, a  $*$ -morphism  $(f, f_\mu^n) : X \rightarrow Y$  is said to be commutative whenever, for every pair  $\mu \leq \mu'$ , one may put  $n = 1$ .

If  $Y = X$ , the identity  $*$ -morphism  $(1_\Lambda, 1_\lambda^n) : X \rightarrow X$  is defined by putting, for each  $\lambda \in \Lambda$  and every  $n \in \mathbb{N}$ ,  $1_\lambda^n \equiv 1_\lambda$

to be the identity  $\mathcal{C}$ -morphism on  $X_\lambda$ . The composition of  $(f, f_\mu^n) : X \rightarrow Y$  with a  $*$ -morphism  $(g, g_\nu^n) : Y \rightarrow Z = (Z_\nu, r_{\nu\nu}, N)$  is defined by

$$(h = fg, h_\nu^n = g_\nu^n f_{g(\nu)}^n) : X \rightarrow Z.$$

The category  $inv^* \mathcal{C}$  is now defined by putting  $Ob(inv^* \mathcal{C}) = Ob(inv \mathcal{C})$  and  $inv^* \mathcal{C}(X, Y)$  to be the set of all  $*$ -morphisms of  $X$  to  $Y$ .

A  $*$ -morphism  $(f, f_\mu^n) : X \rightarrow Y$  is said to be equivalent to a  $*$ -morphism  $(f', f_\mu'^n) : X \rightarrow Y$ , denoted by  $(f, f_\mu^n) \sim (f', f_\mu'^n)$ , if

$$(\forall \mu \in M) (\exists \lambda \in \Lambda, \lambda \geq f(\mu), f'(\mu)) (\exists n \in \mathbb{N}) (\forall n' \geq n)$$

$$f_\mu^n P_{f(\mu)\lambda} = f_\mu'^n P_{f'(\mu)\lambda}.$$

The equality holds for every  $\lambda' \geq \lambda$  as well. The relation  $\sim$  is an equivalence relation on each set  $(inv^* \mathcal{C})(X, Y)$ , and the equivalence class  $[(f, f_\mu^n)]$  of  $(f, f_\mu^n) : X \rightarrow Y$  is briefly denoted by  $f^*$ . The equivalence relation  $\sim$  is compatible with the composition, i.e., if  $(f, f_\mu^n) \sim (f', f_\mu'^n)$  and  $(g, g_\nu^n) \sim (g', g_\nu'^n) : Y \rightarrow Z$ , then

$$(g, g_\nu^n)(f, f_\mu^n) \sim (g', g_\nu'^n)(f', f_\mu'^n) : X \rightarrow Z.$$

The  $pro^*$ -category  $pro^* \mathcal{C}$  is now defined to be the quotient category  $(inv^* \mathcal{C})/\sim$ , i.e.,

$$Ob(pro^* \mathcal{C}) = Ob(inv^* \mathcal{C}) (= Ob(inv \mathcal{C}) = Ob(pro \mathcal{C})),$$

$$(pro^* \mathcal{C})(X, Y) = (inv^* \mathcal{C})(X, Y) / \sim = \{f^* = [(f, f_\mu^n)] \mid (f, f_\mu^n) : X \rightarrow Y\}.$$

Finally, there exists a faithful functor  $\underline{I} : pro \mathcal{C} \rightarrow pro^* \mathcal{C}$ , keeping the objects fixed, such that, for every  $f = [(f, f_\mu^n)] \in (pro \mathcal{C})(XY)$ ,

$$\underline{I}(f) \equiv f^* = [(f, f_\mu^n)] \in (pro^* \mathcal{C})(X, Y),$$

where, for each  $\mu \in M$  and every  $n \in \mathbb{N}$ ,  $f_\mu^n = f_\mu$ .

Let  $\mathcal{D}$  be a full (not essential, but a convenient condition) and pro-reflective subcategory of  $\mathcal{C}$ . Let  $p : X \rightarrow X$  and  $p' : X \rightarrow X'$  be  $\mathcal{D}$ -expansions of the same object  $X$  of  $\mathcal{C}$ , and let  $q : Y \rightarrow Y$  and  $q' : Y \rightarrow Y'$  be  $\mathcal{D}$ -expansions of the same object  $Y$  of  $\mathcal{C}$ . Then there exist two canonical (unique) isomorphisms  $i : X \rightarrow X'$  and  $j : Y \rightarrow Y'$  of  $pro \mathcal{D}$ . Consequently,  $i^* \equiv \underline{I}(i) : X \rightarrow X'$  and  $j^* \equiv \underline{I}(j) : Y \rightarrow Y'$  are isomorphisms of  $pro^* \mathcal{D}$ . A morphism  $f^* : X \rightarrow Y$  is said to be  $pro^* \mathcal{D}$  equivalent to a morphism  $f'^* : X' \rightarrow Y'$ , denoted by  $f^* \sim f'^*$ , if the following diagram in  $pro^* \mathcal{D}$  commutes:

$$\begin{array}{ccc} X & \xrightarrow{i^*} & X' \\ f^* \downarrow & & \downarrow f'^* \\ Y & \xrightarrow{j^*} & Y' \end{array}$$

According to the analogous facts in  $pro\text{-}\mathcal{D}$ , and since  $I$  is a functor, it defines an equivalence relation on the appropriate subclass of  $Mor(pro^*\text{-}\mathcal{D})$ , such that  $f^* \sim f'^*$  and  $g^* \sim g'^*$  imply  $g^*f^* \sim g'^*f'^*$  whenever it is defined. The equivalence class of an  $f^*$  is denoted by  $\langle f^* \rangle$ . Further, given  $p, p', q, q'$  and  $f^*$  as above, there exists a unique  $f'^* (= j^*f^*(i^*)^{-1})$  such that  $f^* \sim f'^*$ . Then the (abstract) coarse shape category  $Sh_{(\mathcal{C}, \mathcal{D})}^*$  for  $(\mathcal{C}, \mathcal{D})$  is defined as follows. The objects of  $Sh_{(\mathcal{C}, \mathcal{D})}^*$  are all the objects of  $\mathcal{C}$ . A morphism  $F^* \in Sh_{(\mathcal{C}, \mathcal{D})}^*(X, Y)$  is the  $(pro^*\text{-}\mathcal{D})$ -equivalence class  $\langle f^* \rangle$  of a morphism  $f^* : X \rightarrow Y$ , with respect to any choice of a pair of  $\mathcal{D}$ -expansions  $p : X \rightarrow X', q : Y \rightarrow Y'$ . In other words, a coarse shape morphism  $F^* : X \rightarrow Y$  is given by a diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ f^* \downarrow & & \downarrow F^* \\ Y & \xrightarrow{q} & Y' \end{array}$$

The composition of an  $F^* : X \rightarrow Y, F^* = \langle f^* \rangle$  and a  $G^* : Y \rightarrow Z, G^* = \langle g^* \rangle$ , is defined by any pair of their representatives, i.e.  $G^*F^* : X \rightarrow Z, G^*F^* = \langle g^*f^* \rangle$ . The identity coarse shape morphism on an object  $X, 1_X^* : X \rightarrow X$ , is the  $(pro^*\text{-}\mathcal{D})$ -equivalence class  $\langle 1_X^* \rangle$  of the identity morphism  $1_X^*$  of  $pro^*\text{-}\mathcal{D}$ .

For every  $\mathcal{C}$ -morphism  $f : X \rightarrow Y$  and every pair of  $\mathcal{D}$ -expansions  $p : X \rightarrow X', q : Y \rightarrow Y'$ , there exists an  $f^* : X \rightarrow Y'$  of  $pro^*\text{-}\mathcal{D}$ , such that the following diagram in  $pro^*\text{-}\mathcal{C}$  commutes:

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ f^* \downarrow & & \downarrow f \\ Y & \xrightarrow{q} & Y' \end{array}$$

(Hereby,  $\mathcal{C} \subseteq pro\text{-}\mathcal{C}$  are considered to be the subcategories of  $pro^*\text{-}\mathcal{C}$ !) The same  $f$  and another pair of  $\mathcal{D}$ -expansions  $p' : X \rightarrow X', q' : Y \rightarrow Y'$  yield an  $f'^* : X' \rightarrow Y'$  in  $pro^*\text{-}\mathcal{D}$ . Then, however,  $f^* \sim f'^*$  in  $pro^*\text{-}\mathcal{D}$  must hold. Thus, every morphism  $f \in \mathcal{C}(X, Y)$  yields a  $(pro^*\text{-}\mathcal{D})$ -equivalence class  $\langle f^* \rangle$ , i.e., a coarse shape morphism  $F^* \in Sh_{(\mathcal{C}, \mathcal{D})}^*(X, Y)$ . Therefore, by putting  $S^*(X) = X, X \in Ob\mathcal{C}$ , and  $S^*(f) = F^* = \langle f^* \rangle, f \in Mor\mathcal{C}$ , a unique functor

$$S_{(\mathcal{C}, \mathcal{D})}^* : \mathcal{C} \rightarrow Sh_{(\mathcal{C}, \mathcal{D})}^*,$$

called the abstract coarse shape functor, is defined. Moreover, the functor  $S_{(\mathcal{C}, \mathcal{D})}^*$  factorizes as  $S_{(\mathcal{C}, \mathcal{D})}^* = I_{(\mathcal{C}, \mathcal{D})} S_{(\mathcal{C}, \mathcal{D})}$ , where  $S_{(\mathcal{C}, \mathcal{D})} : \mathcal{C} \rightarrow Sh_{(\mathcal{C}, \mathcal{D})}$  is the abstract shape functor, while  $I_{(\mathcal{C}, \mathcal{D})} : Sh_{(\mathcal{C}, \mathcal{D})} \rightarrow Sh_{(\mathcal{C}, \mathcal{D})}^*$  is induced by the “inclusion” functor  $I \equiv I_{\mathcal{D}} : pro\text{-}\mathcal{D} \rightarrow pro^*\text{-}\mathcal{D}$ .

As in the case of the abstract shape, the most interesting example of the above construction is  $\mathcal{C} = HTop$ -the homotopy category of topological spaces and  $\mathcal{D} = HPol$ -the homotopy category of polyhedra (or  $\mathcal{D} = HANR$ -the

homotopy category of ANR's for metric spaces). In this case, one speaks about the (ordinary or standard) coarse shape category

$$Sh_{(HTop,HPol)}^* \equiv Sh^*(Top) \equiv Sh^*$$

of topological spaces and of (ordinary or standard) coarse shape functor

$$S^* : HTop \rightarrow Sh^*,$$

which factorizes as  $S^* = IS$ , where  $S : HTop \rightarrow Sh$  is the shape functor, and  $I : Sh \rightarrow Sh^*$  is induced by the “inclusion” functor

$$\underline{I} \equiv \underline{I}_{HPol} : Ipro - HPol \rightarrow pro^* - HPol.$$

The realizing category for  $Sh^*$  is the category  $pro^* - HPol$  (or  $pro^* - HANR$ ). The underlying theory might be called the (ordinary or standard) coarse shape theory (for topological spaces). Clearly, on locally nice spaces (polyhedra, CW-complexes, ANR's, ...) the coarse shape type classification coincides with the shape type classification and, consequently, with the homotopy type classification. However, in general (even for metrizable continua), the shape type classification is strictly coarser than the homotopy type classification, and the coarse shape type classification is strictly coarser than the shape type classification.

### 3. Enriched pro-categories

Given a category  $\mathcal{C}$ , we are going to construct a class of categories having the same objects-all inverse systems in the category  $\mathcal{C}$ -by enriching the morphism sets such that  $pro - \mathcal{C}$  and  $pro^* - \mathcal{C}$  become the very special cases of these new categories, so called enriched  $pro$ -categories.

**Definition 1.** Let  $\mathcal{C}$  be a category, let  $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$  and  $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$  be inverse systems in  $\mathcal{C}$  and let  $J = (J, \leq)$  be a directed partially ordered set. A  $J$ -morphism (of  $\mathbf{X}$  to  $\mathbf{Y}$  in  $\mathcal{C}$ ) is every triple  $(\mathbf{X}, (f, (f_\mu^j)), \mathbf{Y})$ , denoted by  $(f, f_\mu^j) : \mathbf{X} \rightarrow \mathbf{Y}$ , where  $(f, f_\mu^j)$  is an ordered pair consisting of a function  $f : M \rightarrow \Lambda$ , called the index function, and, for each  $\mu \in M$ , of a family  $(f_\mu^j)$  of  $\mathcal{C}$ -morphisms  $f_\mu^j : X_{f(\mu)} \rightarrow Y_\mu$ ,  $j \in J$ , such that, for every related pair  $\mu \leq \mu'$  in  $M$ , there exists a  $\lambda \in \Lambda$ ,  $\lambda \geq f(\mu)$ ,  $f(\mu')$ , and there exists a  $j \in J$  so that, for every  $j' \geq j$ ,

$$f_\mu^{j'} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{j'} p_{f(\mu')\lambda}.$$

If the index function  $f$  is increasing and, for every pair  $\mu \leq \mu'$ , one may put  $\lambda \geq f(\mu')$ , then  $(f, f_\mu^j)$  is said to be a simple  $J$ -morphism. If, in addition,  $M = \Lambda$  and  $f = 1_\Lambda$ , then  $(1_\Lambda, f_\lambda^j)$  is said to be a level  $J$ -morphism. Further, if the equality holds for every  $j \in J$ , then  $(f, f_\mu^j) : \mathbf{X} \rightarrow \mathbf{Y}$  is said to be a commutative  $J$ -morphism. (If there exists  $\min J \equiv j_*$ , the commutativity means that one may put  $j = j_*$ .)

**Remark 1.** The equality condition of Definition 1 obviously holds for every  $\lambda' \geq \lambda$  as well. Every commutative  $J$ -morphism of inverse systems  $(f, f_\mu^j) : \mathbf{X} \rightarrow \mathbf{Y}$  yields a family of morphisms  $(f^j = f, f_\mu^j) : \mathbf{X} \rightarrow \mathbf{Y}$ ,  $j \in J$ , of  $inv - \mathcal{C}$ . On the other side, every family of simple morphisms  $(f^j, f_\mu^j) : \mathbf{X} \rightarrow \mathbf{Y}$ ,  $j \in J$ , of  $inv - \mathcal{C}$ , such that  $f^j = f$  for all  $j$ , determines the unique commutative  $J$ -morphism of the inverse systems  $(f, f_\mu^j) : \mathbf{X} \rightarrow \mathbf{Y}$ . This indicates the significant difference between (a huge generalization of) the standard morphisms of inverse systems comparing to the new  $J$ -morphisms.

**Lemma 1.** Let  $(f, f_\mu^j) : \mathbf{X} \rightarrow \mathbf{Y}$  and  $(g, g_\nu^j) : \mathbf{Y} \rightarrow \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$  be  $J$ -morphisms (of inverse systems in a category  $\mathcal{C}$ ).

Then  $(h, h_v^j)$ , where  $h = fg$  and  $h_v^j = g_v^j f_{g(v)}^j$ ,  $j \in J, v \in N$ , is a  $J$ -morphism of  $X$  to  $Z$ .

**Proof.** Let  $v, v' \in N, v \leq v'$ , be given. Since  $(g, g_v^j)$  is a  $J$ -morphism, there exists a  $\mu \in M, \mu \geq g(v), g(v')$ , and there exists a  $j_0 \in J$  such that, for every  $j' \geq j_0$ ,

$$g_v^{j'} q_{g(v)\mu} = r_{vv'} g_v^{j'} q_{g(v')\mu}.$$

Since  $(f, f_\mu^j)$  is a  $J$ -morphism, for the pair  $g(v) \leq \mu$ , there exist a  $\lambda_1 \geq fg(v), f(\mu)$  in  $\Lambda$  and a  $j_1 \in J$  such that, for every  $j' \geq j_1$ ,

$$f_{g(v)}^{j'} p_{fg(v)\lambda_1} = q_{g(v)\mu} f_\mu^{j'} p_{f(\mu)\lambda_1}.$$

Further, for the pair  $g(v') \leq \mu$ , there exist a  $\lambda_2 \geq fg(v'), f(\mu)$  in  $\Lambda$  and a  $j_2 \in J$  such that, for every  $j' \geq j_2$ ,

$$f_{g(v')}^{j'} p_{fg(v')\lambda_2} = q_{g(v')\mu} f_\mu^{j'} p_{f(\mu)\lambda_2}.$$

Since  $\Lambda$  and  $J$  are directed, there exist  $\lambda \in \Lambda, \lambda \geq \lambda_1, \lambda_2$ , and a  $j \in J, j \geq j_0, j_1, j_2$ , respectively. Then, for every  $j' \geq j$ , one straight forwardly establishes

$$g_v^{j'} f_{g(v)}^{j'} p_{fg(v)\lambda} = r_{vv'} g_v^{j'} f_{g(v')}^{j'} p_{fg(v')\lambda},$$

which proves that  $(h = fg, h_v^j = g_v^j f_{g(v)}^j) : X \rightarrow Z$  is a  $J$ -morphism.  $\square$

Lemma 1 enables us to define the composition of  $J$ -morphisms of inverse systems: If  $(f, f_\mu^j) : X \rightarrow Y$  and  $(g, g_v^j) : Y \rightarrow Z$ , then  $(g, g_v^j)(f, f_\mu^j) = (h, h_v^j) : X \rightarrow Z$ , where  $h = fg$  and  $h_v^j = g_v^j f_{g(v)}^j$ . Clearly, this composition is associative.

**Lemma 2.** The composition of commutative  $J$ -morphisms of inverse systems in  $\mathcal{C}$  is a commutative  $J$ -morphism.

**Proof.** It is a straightforward consequence of the defining coordinatewise (by  $j \in J$ ) composition.

Given an inverse system  $X = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$  in  $\mathcal{C}$ , let  $(1_\Lambda, 1_{X_\lambda}^j)$ , consists of the identity function  $1_\Lambda$  and, for each  $\lambda \in \Lambda$ , of the family induced by the same identity morphism  $1_{X_\lambda}^j = 1_{X_\lambda}, j \in J$ , of  $\mathcal{C}$ . Then  $(1_\Lambda, 1_{X_\lambda}^j) : X \rightarrow X$  is a  $J$ -morphism (commutative and leveled). One readily sees that, for every  $(f, f_\mu^j) : X \rightarrow Y$  and every  $(g, g_\lambda^j) : Z \rightarrow X, (f, f_\mu^j)(1_\Lambda, 1_{X_\lambda}^j) = (f, f_\mu^j)$  and  $(1_\Lambda, 1_{X_\lambda}^j)(g, g_\lambda^j) = (g, g_\lambda^j)$  hold. Thus,  $(1_\Lambda, 1_{X_\lambda}^j)$  may be called the identity  $J$ -morphism on  $X$ .

By summarizing, for every category  $\mathcal{C}$  and every directed partially ordered set  $J$ , there exists a category, denoted by  $inv^J\mathcal{C}$ , consisting of the object class  $Ob(inv^J\mathcal{C}) = Ob(inv\mathcal{C})$  and of the morphism class  $Mor(inv^J\mathcal{C})$  of all the sets  $(inv^J\mathcal{C})(X, Y)$  of all  $J$ -morphisms  $(f, f_\mu^j)$  of  $X$  to  $Y$ , endowed with the composition and identities described above. By Lemma 2., there exists a subcategory  $(inv^J\mathcal{C})_c$  of  $inv^J\mathcal{C}$  with the same object class and with the morphism class  $Mor(inv^J\mathcal{C})_c$  consisting of all commutative  $J$ -morphisms of inverse systems in  $\mathcal{C}$ .

Let us now define an appropriate equivalence relation on each set  $(inv^J\mathcal{C})(X, Y)$ .

**Definition 2.** A  $J$ -morphism  $(f, f_\mu^j) : X \rightarrow Y$  of inverse systems in  $\mathcal{C}$  is said to be equivalent to a  $J$ -morphism  $(f', f'_\mu^j) : X \rightarrow Y$ , denoted by  $(f, f_\mu^j) \sim (f', f'_\mu^j)$ , if every  $\mu \in M$  admits a  $\lambda \in \Lambda, \lambda \geq f(\mu), f'(\mu)$ , and a  $j \in J$  such that, for every  $j' \geq j$ ,

$$f_\mu^{j'} p_{f(\mu)\lambda} = f'_\mu^{j'} p_{f'(\mu)\lambda}.$$

**Lemma 3.** The defining equality holds for every  $\lambda' \geq \lambda$  as well, and the relation  $\sim$  is an equivalence relation on each set  $(\text{inv}^J\text{-}\mathcal{C})(X, Y)$ . The equivalence class  $[(f, f_\mu^j)]$  of a  $J$ -morphism  $(f, f_\mu^j) : X \rightarrow Y$  is briefly denoted by  $f^j \equiv f$ .

**Proof.** The first claim is trivial. The relation  $\sim$  is obviously reflexive and symmetric. To prove transitivity, let, for a given  $\mu \in M$ , the indices  $\lambda_1$  and  $j_1$  realize the first relation,  $(f, f_\mu^j) \sim (f', f_\mu^{j'})$ , and the indices  $\lambda_2$  and  $j_2$ -the second one- $(f', f_\mu^{j'}) \sim (f'', f_\mu^{j''})$ . Since  $\Lambda$  and  $J$  are directed, there exist a  $\lambda \geq \lambda_1, \lambda_2$  and a  $j \geq j_1, j_2$  respectively, that realize transitivity,  $(f, f_\mu^j) \sim (f'', f_\mu^{j''})$ .

**Lemma 4.** Let  $(f, f_\mu^j), (f', f_\mu^{j'}) : X \rightarrow Y$  and  $(g, g_\nu^j), (g', g_\nu^{j'}) : Y \rightarrow Z$  be  $J$ -morphisms of inverse systems in  $\mathcal{C}$ . If  $(f, f_\mu^j) \sim (f', f_\mu^{j'})$  and  $(g, g_\nu^j)(f, f_\mu^j) \sim (g', g_\nu^{j'})(f', f_\mu^{j'})$ , then  $(g, g_\nu^j)(f, f_\mu^j) \sim (g', g_\nu^{j'})(f', f_\mu^{j'})$ .

**Proof.** According to Lemma 3. (transitivity), it suffices to prove that  $(g, g_\nu^j)(f, f_\mu^j) \sim (g, g_\nu^j)(f', f_\mu^{j'})$  and  $(g, g_\nu^j)(f, f_\mu^j) \sim (g', g_\nu^{j'})(f, f_\mu^j)$ . Given a  $\nu \in N$ , choose a  $\lambda \in \Lambda$ ,  $\lambda \geq fg(\nu), f'g(\nu)$ , and a  $j \in J$ , by  $(f, f_\mu^j) \sim (f', f_\mu^{j'})$  for  $\mu = g(\nu)$ . Then, for every  $j' \geq j$ ,

$$g_\nu^j f_{g(\nu)}^j p_{fg(\nu)\lambda} = g_\nu^{j'} f_{g(\nu)}^{j'} p_{f'g(\nu)\lambda}$$

Thus,  $(g, g_\nu^j)(f, f_\mu^j) \sim (g, g_\nu^j)(f', f_\mu^{j'})$ . Further, if  $(g, g_\nu^j) \sim (g', g_\nu^{j'})$ , then for a given  $\nu \in N$  there exist a  $\mu \geq g(\nu), g'(\nu)$  and a  $j_1$  such that

$$g_\nu^j q_{g(\nu)\mu} = g_\nu^{j'} q_{g'(\nu)\mu}$$

whenever  $j' \geq j_1$ . Since  $(f, f_\mu^j)$  is a  $J$ -morphism, there exist a  $\lambda_1 \geq fg(\nu), f(\mu)$  and a  $j_2$  such that, for every  $j' \geq j_2$ ,

$$f_{g(\nu)}^j p_{fg(\nu)\lambda_1} = q_{g(\nu)\mu} f_\mu^{j'} p_{f(\mu)\lambda_1}$$

In the same way, there exist a  $\lambda_2 \geq f'g'(\nu), f(\mu)$  and a  $j_3$  such that, for every  $j' \geq j_3$ ,

$$f_{g'(\nu)}^{j'} p_{f'g'(\nu)\lambda_2} = q_{g'(\nu)\mu} f_\mu^{j'} p_{f(\mu)\lambda_2}$$

Since  $\Lambda$  and  $J$  are directed, there exist a  $\lambda \geq \lambda_1, \lambda_2$ , and  $j \geq j_1, j_2, j_3$  respectively. Then, for every  $j' \geq j$ ,

$$g_\nu^j f_{g(\nu)}^j p_{fg(\nu)\lambda} = g_\nu^{j'} f_{g'(\nu)}^{j'} p_{f'g'(\nu)\lambda}$$

Therefore,  $(g, g_\nu^j)(f, f_\mu^j) \sim (g', g_\nu^{j'})(f, f_\mu^j)$ . □

By Lemmata 3 and 4 one may compose the equivalence classes of  $J$ -morphisms of inverse systems in  $\mathcal{C}$  by means of any pair of their representatives, i.e.,  $\mathbf{gf} = \mathbf{h} \equiv [(h, h_\nu^j)]$ , where  $(h, h_\nu^j) = (g, g_\nu^j)(f, f_\mu^j) = (fg, g_\nu^j f_\mu^j)$ . The corresponding quotient category  $(\text{inv}^J\text{-}\mathcal{C})/\sim$  is denoted by  $\text{pro}^J\text{-}\mathcal{C}$ . There exists a subcategory  $(\text{pro}^J\text{-}\mathcal{C})_c \subseteq \text{pro}^J\text{-}\mathcal{C}$  determined by all equivalence classes having commutative representatives. Clearly,  $(\text{pro}^J\text{-}\mathcal{C})_c$  is isomorphic to the quotient category  $(\text{inv}^J\text{-}\mathcal{C})_c/\sim$ . Further, one may consider  $\text{pro}\text{-}\mathcal{C} = (\text{inv}\text{-}\mathcal{C})/\sim$  as a subcategory of  $(\text{pro}^J\text{-}\mathcal{C})_c$  and, consequently, as a subcategory of  $\text{pro}^J\text{-}\mathcal{C}$  (see also Theorem 1 below). First, recall the well known lemma (see [13], Lemma I. 1.1):

**Lemma 5.** Let  $(\Lambda, \leq)$  be a directed set and let  $(M, \leq)$  be a cofinite directed set. Then every function  $f : M \rightarrow \Lambda$  admits an increasing function  $f' : M \rightarrow \Lambda$  such that  $f \leq f'$ .

**Lemma 6.** Let  $X = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$  and  $Y = (Y_\mu, q_{\mu\mu'}, M)$  be inverse systems in  $\mathcal{C}$  with  $M$  cofinite. Then every morphism  $f = [(f, f_\mu^j)] : X \rightarrow Y$  of  $pro^J\text{-}\mathcal{C}$  admits a simple representative  $(f', f_{\mu'}^j) : X \rightarrow Y$ .

**Proof.** Let  $\mu \in M$ . If  $\mu$  has no predecessors, choose any  $\lambda \in \Lambda$ ,  $\lambda \geq f(\mu)$ , and put  $\varphi(\mu) = \lambda$ . If  $\mu$  is not an initial element of  $M$ , let  $\mu_1, \dots, \mu_m \in M$ ,  $m \in \mathbb{N}$ , be all the predecessors of  $\mu$  ( $M$  is cofinite). Since  $(f, f_\mu^j)$  is a  $J$ -morphism, for every  $i \in \{1, \dots, m\}$  and every pair  $\mu_i \leq \mu$ , there exists a  $\lambda_i \in \Lambda$ ,  $\lambda_i \geq f(\mu_i)$ ,  $f(\mu)$ , and there exists a  $j_i \in J$ , such that, for every  $j' \geq j_i$ , the appropriate condition holds. Choose any  $\lambda \in \Lambda$ ,  $\lambda \geq \lambda_i$  for all  $i \in \{1, \dots, m\}$  ( $\Lambda$  is directed), and put  $\varphi(\mu) = \lambda$ . This defines a function  $\varphi : M \rightarrow \Lambda$ . Notice that  $f \leq \varphi$ . By Lemma 5., there exists an increasing function  $f' : M \rightarrow \Lambda$  such that  $\varphi \leq f'$ . Hence,  $f \leq f'$ . Now, for every  $\mu \in M$ , put  $f_{\mu'}^j = f_\mu^j p_{f(\mu), f'(\mu)}$ . One readily verifies that  $(f', f_{\mu'}^j) : X \rightarrow Y$  is a simple  $J$ -morphism and that  $(f', f_{\mu'}^j) \sim (f, f_\mu^j)$ .

Let us define a certain functor  $\underline{I} \equiv \underline{I}_C^J : pro\text{-}\mathcal{C} \rightarrow pro^J\text{-}\mathcal{C}$ . Put  $\underline{I}(X) = X$ , for every inverse system  $X$  in  $\mathcal{C}$ . If  $f \in pro\text{-}\mathcal{C}(X, Y)$  and if  $(f, f_\mu)$  is a representative of  $f$ , put

$$\underline{I}(f) = f^J = [(f, f_\mu^j)] \in (pro^J\text{-}\mathcal{C})(X, Y),$$

where  $(f, f_\mu^j)$  is induced by  $(f, f_\mu)$ , i.e., for each  $\mu \in M$ ,  $f_\mu^j = f_\mu$  for all  $j \in J$ . One straightforwardly verifies that  $\underline{I}(f)$  is well defined. Notice that every induced  $J$ -morphism is commutative. Therefore,  $\underline{I}$  is a functor of  $pro\text{-}\mathcal{C}$  to the subcategory  $(pro^J\text{-}\mathcal{C})_c \subseteq pro^J\text{-}\mathcal{C}$ .

**Theorem 1.** The functor  $\underline{I} : pro\text{-}\mathcal{C} \rightarrow (pro^J\text{-}\mathcal{C})_c \subseteq pro^J\text{-}\mathcal{C}$  is faithful.

**Proof.** The functoriality follows straightforwardly. Let  $f^J = \underline{I}(f) = \underline{I}(f')$  and  $f'^J = f'^J$ . Let  $(f, f_\mu)$  and  $(f', f_{\mu'})$  be any representatives of  $f$  and  $f'$  respectively. By definition of the functor  $\underline{I}$ ,  $f^J = [(f, f_\mu^j = f_\mu)]$  and  $f'^J = [(f', f_{\mu'}^j = f_{\mu'})]$ . Since  $(f, f_\mu^j) \sim (f', f_{\mu'}^j)$ , for every  $\mu \in M$ , there exist a  $\lambda \geq f(\mu)$ ,  $f'(\mu)$  and a  $j$  such that, for every  $j' \geq j$ ,

$$f_\mu^j p_{f(\mu)\lambda} = f_{\mu'}^j p_{f'(\mu)\lambda}.$$

This means that

$$f_\mu p_{f(\mu)\lambda} = f_{\mu'} p_{f'(\mu)\lambda}$$

holds. Therefore,  $(f, f_\mu) \sim (f', f_{\mu'})$ , i.e.,  $f = f'$ . □

**Remark 2.** The functor  $\underline{I}$  is not full. For instance, let us consider the restriction  $(pro\text{-}\mathcal{C})(X, T) \rightarrow (pro^J\text{-}\mathcal{C})_c(X, T)$ , where  $T = (T_0 \equiv T)$  is a rudimentary inverse system. Let  $f \in (pro\text{-}\mathcal{C})(X, T)$ . Then every representative  $(f, f_0)$  of  $f$  is uniquely determined by a  $\lambda_0 \in \Lambda$  ( $f(0) = \lambda_0$ ) and by a morphism  $f_0 \equiv f_{\lambda_0} \in \mathcal{C}(X_{\lambda_0}, T)$ . However, it is not the case for an  $f^J \in (pro^J\text{-}\mathcal{C})_c(X, T)$ . Indeed, if  $(f, f_0^j)$  is a representative of  $f^J$ , then  $f(0) = \lambda_0 \in \Lambda$ , while  $(f_0^j \equiv f_{\lambda_0}^j)_{j \in J}$  is a family of morphisms  $f_{\lambda_0}^j \in \mathcal{C}(X_{\lambda_0}, T)$ . Notice that  $(f, f_0^j) \sim (f', f_0'^j)$  if and only if

$$(\exists \lambda \geq \lambda_0, \lambda_0') (\exists j) (\forall j' \geq j) f_0^j p_{\lambda_0 \lambda} = f_0'^j p_{\lambda_0' \lambda}.$$

By the well known “Mardešić trick”, every inverse system  $X$  in  $\mathcal{C}$  is isomorphic (in  $pro\text{-}\mathcal{C}$ ) to a cofinite inverse system  $X'$ . If  $f : X \rightarrow X'$  is an isomorphism of  $pro\text{-}\mathcal{C}$ , then  $\underline{I}(f) : X \rightarrow X'$  is an isomorphism of  $pro^J\text{-}\mathcal{C}$ . Therefore, the next corollary holds.



**Corollary 1.** Every inverse system  $X$  in  $\mathcal{C}$  is isomorphic in  $pro^J\text{-}\mathcal{C}$  to a cofinite inverse system  $X'$ .

A morphism  $f : X \rightarrow Y$  of  $pro^J\text{-}\mathcal{C}$  does not admit, in general, a level representative. However, the following “reindexing theorem” will help to overcome some technical difficulties concerning this fact.

**Theorem 2.** Let  $f \in (pro^J\text{-}\mathcal{C})(X, Y)$ . Then there exist inverse systems  $X'$  and  $Y'$  in  $\mathcal{C}$  having the same cofinite index set  $(N, \leq)$ , there exists a morphism  $f' : X' \rightarrow Y'$  having a level representative  $(1_N, f'_v)$  and there exist isomorphisms  $i : X \rightarrow X'$  and  $j : Y \rightarrow Y'$  of  $pro^J\text{-}\mathcal{C}$ , such that the following diagram in  $pro^J\text{-}\mathcal{C}$  commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & & \downarrow j \\ X' & \xrightarrow{f'} & Y' \end{array}$$

**Proof.** Let  $f \in (pro^J\text{-}\mathcal{C})(X, Y)$ . By Corollary 1, there exist cofinite inverse systems  $\tilde{X} = (\tilde{X}_\alpha, \tilde{p}_{\alpha\alpha'}, A)$  and  $\tilde{Y} = (\tilde{Y}_\beta, \tilde{q}_{\beta\beta'}, B)$ , and there exist isomorphisms  $u : X \rightarrow \tilde{X}$  and  $v : Y \rightarrow \tilde{Y}$  of  $pro^J\text{-}\mathcal{C}$ . Let  $\tilde{f} = vfu^{-1} : \tilde{X} \rightarrow \tilde{Y}$ . By Lemma 6, there exists a simple representative  $(w, w_\beta^j)$  of  $\tilde{f}$ . Let

$$N = \{v \equiv (\alpha, \beta) \mid \alpha \in A, \beta \in B, w(\beta) \leq \alpha\} \subseteq A \times B,$$

and define  $(N, \leq)$  coordinatewise, i.e.,  $v = (\alpha, \beta) \leq (\alpha', \beta') = v'$  if and only if  $\alpha \leq \alpha'$  in  $A$  and  $\beta \leq \beta'$  in  $B$ . Clearly,  $N$  is preordered. Let any  $v = (\alpha, \beta), v' = (\alpha', \beta') \in N$  be given. Since  $B$  is directed, there exists a  $\beta_0 \geq \beta, \beta'$ . Since  $A$  is directed, there exists an  $\alpha_0 \geq \alpha, \alpha'$ ,  $w(\beta_0)$ . Then  $(\alpha_0, \beta_0) \equiv v_0 \in N$  and  $v_0 \geq v, v'$ . Thus,  $N$  is directed. Further, since  $A$  and  $B$  are cofinite and since  $N \subseteq A \times B$  is (pre)ordered coordinatewise, the set  $N$  is cofinite too. Let us now construct desired inverse systems  $X' = (X'_v, p'_{vv'}, N)$  and  $Y' = (Y'_v, q'_{vv'}, N)$ . Given a  $v = (\alpha, \beta) \in N$ , put  $X'_v = \tilde{X}_\alpha$  and  $Y'_v = \tilde{Y}_\beta$ . For every related pair  $v = (\alpha, \beta) \leq (\alpha', \beta') = v'$  in  $N$ , put  $p'_{vv'} = \tilde{p}_{\alpha\alpha'}$  and  $q'_{vv'} = \tilde{q}_{\beta\beta'}$ . Now, for each  $v = (\alpha, \beta) \in N$  and every  $j \in J$ , put  $f_v^{j,j} = w_\beta^j \tilde{p}_{w(\beta)\alpha} : X'_v \rightarrow Y'_v$ . Then  $(1_N, f_v^{j,j}) : X' \rightarrow Y'$  is a simple  $J$ -morphism. Indeed, if  $v \leq v'$ , then  $\beta \leq \beta'$ . Since  $(w, w_\beta^j)$  is simple, there exists a  $j \in J$  such that, for every  $j' \geq j$ ,

$$w_\beta^j \tilde{p}_{w(\beta)w(\beta')} = \tilde{q}_{\beta\beta'} w_{\beta'}^{j'}.$$

Since  $\alpha \geq w(\beta)$ ,  $\alpha' \geq w(\beta')$ ,  $w(\beta') \geq w(\beta)$  and  $\alpha' \geq \alpha$ , it implies that

$$f_v^{j,j} p'_{vv'} = w_\beta^j \tilde{p}_{w(\beta)\alpha} \tilde{p}_{\alpha\alpha'} = w_\beta^j \tilde{p}_{w(\beta)w(\beta')} \tilde{p}_{w(\beta')\alpha'} = \tilde{q}_{\beta\beta'} w_{\beta'}^{j'} \tilde{p}_{w(\beta')\alpha'} = q'_{vv'} f_{v'}^{j',j'}.$$

Let  $s : N \rightarrow \Lambda$  be defined by putting  $s(v) = \alpha$ , where  $v = (\alpha, \beta)$ , and let, for each  $v \in N$  and every  $j \in J$ ,  $s_v^j : \tilde{X}_\alpha \rightarrow X'_v = \tilde{X}_\alpha$  be the identity  $1_{\tilde{X}_\alpha}$  of  $\mathcal{C}$ . In the same way, let  $t : N \rightarrow M$  be defined by putting  $t(v) = \beta$ , and let, for each  $v \in N$  and every  $j \in J$ ,  $t_v^j : \tilde{Y}_\beta \rightarrow Y'_v = \tilde{Y}_\beta$  be the identity  $1_{\tilde{Y}_\beta}$  of  $\mathcal{C}$ . It is readily seen that  $s = [(s, s_v^j)] : \tilde{X} \rightarrow X'$  and  $t = [(t, t_v^j)] : \tilde{Y} \rightarrow Y'$  are simple commutative morphisms of  $pro^J\text{-}\mathcal{C}$ . Even more, they are induced by morphisms  $(s, s_v = 1_{\tilde{X}_\alpha})$  and  $(t, t_v = 1_{\tilde{Y}_\beta})$  of  $inv\text{-}\mathcal{C}$  respectively. Notice that, in  $pro\text{-}\mathcal{C}$ ,  $[(s, s_v)] : \tilde{X} \rightarrow X'$  and  $[(t, t_v)] : \tilde{Y} \rightarrow Y'$  are isomorphisms. Since  $s = \underline{I}[(s, s_v)]$  and  $t = \underline{I}[(t, t_v)]$ , we infer that  $s$  and  $t$  are isomorphisms of  $pro^J\text{-}\mathcal{C}$ . Moreover, for every  $v = (\alpha, \beta) \in N$  and every  $j \in J$ ,

$$t_v^j w_{t(v)}^j \tilde{P}_{wt(v)\alpha} = w_\beta^j \tilde{P}_{w(\beta)\alpha} = f_v'^j = f_v'^j s_v^j,$$

which implies that

$$(t, t_v^j)(w, w_\beta^j) \sim (1_N, f_v'^j)(s, s_v^j).$$

Therefore,  $t\tilde{f} = f's$ . Finally, put  $i \equiv su : X \rightarrow X'$  and  $j \equiv tv : Y \rightarrow Y'$ , which are isomorphisms of  $pro^J\text{-C}$ . Then

$$jf = tvf = t\tilde{f}u = f'su = f'i,$$

that completes the proof of the theorem. □

**Theorem 3.** Let  $\mathcal{C}$  be a category. Then

(i)  $pro\text{-C} = pro^{(1)}\text{-C}$ ;

(ii)  $pro^*\text{-C} = pro^{\mathbb{N}}\text{-C}$ ;

(iii) If  $J$  is a directed partially ordered set having  $\max J$ , then  $pro^J\text{-C} \cong pro\text{-C}$ .

Especially, the “inclusion” functor  $I : pro\text{-C} \rightarrow pro^J\text{-C}$  is a category isomorphism;

(iv) If  $J$  and  $K$  are finite directed partially ordered sets, then one may identify  $pro^J\text{-C} \cong pro^K\text{-C} \cong pro\text{-C}$ .

(v) If there exists  $\max J$ , then, for every  $L$ , there exists the canonical inclusion functor  $\underline{I} : pro^J\text{-C} \rightarrow pro^L\text{-C}$  keeping the objects fixed.

**Proof.** Statements (i) and (ii) are obviously true by the definition of  $pro^J\text{-C}$ . In order to prove (iii), it suffices to show that the functor  $I : pro\text{-C} \rightarrow pro^J\text{-C}$  is fully faithful, i.e., that every

$$f = [(f, f_\mu^j)] : X = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \rightarrow (Y_\mu, q_{\mu\mu'}, M) = Y$$

of  $pro^J\text{-C}$  is uniquely determined by

$$(f, f_\mu^{j^*}) : X \rightarrow Y, j^* \equiv \max J,$$

which belongs to  $(inv\text{-C})(X, Y)$ . Indeed, since  $\max J \equiv j^*$  exists, Definition 1 implies that

$$(\forall \mu \leq \mu') (\exists \lambda \geq f(\mu), f(\mu'))$$

$$f_\mu^{j^*} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{j^*} p_{f(\mu')\lambda}.$$

This means that

$$(f, f_\mu^{j^*}) : X \rightarrow Y$$

is a morphism of  $inv\text{-C}$ . Further, if

$$(f', f'_\mu^j): X \rightarrow Y$$

is an arbitrary representative of  $f$ , then

$$(f', f'_\mu^{j^*}): X \rightarrow Y$$

belongs to  $(inv-C)(X, Y)$  as well and, moreover,  $(f', f'_\mu^{j^*}) \sim (f, f_\mu^{j^*})$  in  $inv-C$  is equivalent to  $(f', f'_\mu^j) \sim (f, f_\mu^j)$  in  $inv^j-C$ . The conclusion follows. Statement (iv) is an immediate consequence of (iii) because every such finite set must have a unique maximal element. Statement (v) follows by (iv) because every  $f = [(f, f'_\mu^j)] \in (pro^j-C)(X, Y)$  is determined by  $(f, f_\mu^{\max J}) \in (inv-C)(X, Y)$ , which induces a unique  $f' = [(f' = f, f'_\mu^l = f_\mu^{\max J})] \in (pro^L-C)_C(X, Y) \subseteq pro^L-C(X, Y)$ .  $\square$

According to Theorem 3, only a  $(J, \leq)$  having no maximal element is interesting because the existence of  $\max J$  turns us back to the “trivial” case of  $pro-C$ . In order to relate  $pro^J-C$  to a  $pro^K-C$  in a “nontrivial” case, we have established the following fact only.

**Theorem 4.** Let  $\mathcal{C}$  be a category, let  $J$  be a well ordered set and let  $K$  be a directed partially ordered set, both without maximal elements. If there exists an increasing function  $\phi : J \rightarrow K$  such that  $\phi[J]$  is cofinal in  $K$ , then there exists a functor

$$\underline{T} : pro^J-C \rightarrow pro^K-C$$

keeping the objects fixed, and  $T$  does not depend on  $\phi$ . Furthermore, for every pair  $X, Y$  of inverse systems in  $\mathcal{C}$ , the equivalence

$$(X \cong Y \text{ in } pro^J-C) \Leftrightarrow (X \cong Y \text{ in } pro^K-C)$$

holds true.

**Proof.** Since  $\phi : J \rightarrow K$  is cofinal, for each  $k \in K$ , the subset

$$J_k = \{j \mid k \leq \phi(j)\} \subseteq J$$

is not empty. Since  $J$  is well ordered, there exists  $\min J_k$ . Furthermore,

$$k \leq k' \Rightarrow j_k \equiv \min J_k \leq \min J_{k'} \equiv j_{k'}$$

because  $\phi$  is increasing. Given an

$$f = [(f, f'_\mu^j)]: X = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \rightarrow (Y_\mu, q_{\mu\mu'}, M) = Y$$

of  $pro^J-C$ , put

$$f' = f : M \rightarrow \Lambda$$

and

$$(\forall \mu \in M)(\forall k \in K) f'_\mu{}^k = f_\mu^{j_k} : X_{f'(\mu)} \rightarrow Y_\mu.$$

Then

$$(f', f'_\mu{}^k) : X \rightarrow Y$$

is a morphism of  $inv^K\text{-C}$ . Indeed, since  $(f, f_\mu^j)$  is a morphism of  $inv^J\text{-C}$ , given a related pair  $\mu \leq \mu'$ , there exist a  $\lambda \geq f(\mu)$ ,  $f(\mu')$  and a  $j$  such that, for every  $j' \geq j$ ,

$$f_\mu^{j'} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{j'} p_{f(\mu')\lambda}.$$

Choose  $k = \phi(j)$ , and let  $k' \geq k$ . Then  $j_{k'} \geq j_k$  and

$$f'_\mu{}^{k'} p_{f'(\mu)\lambda} = f_\mu^{j_{k'}} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{j_{k'}} p_{f(\mu')\lambda} = q_{\mu\mu'} f_{\mu'}^{k'} p_{f(\mu')\lambda}.$$

that proves the claim. Denote

$$f' = [(f', f'_\mu{}^k)] : X \rightarrow Y$$

which is a morphism of  $pro^K\text{-C}$ . Now a straightforward verification shows that the assignments

$$X \mapsto \underline{T}(X) = X, f \mapsto \underline{T}(f) = f'$$

define a functor

$$\underline{T} : pro^J\text{-C} \rightarrow pro^K\text{-C}$$

Finally, if  $\psi : J \rightarrow K$  has the same properties as  $\phi$ , then one readily sees that  $(f'', f''_\mu{}^k) : X \rightarrow Y$ , constructed by means of  $\psi$ , is equivalent to  $(f', f'_\mu{}^k)$  in  $inv^K\text{-C}$ . Thus,  $\underline{T}$  does not depend on any such particular function. In order to prove the last statement, firstly notice that the implication

$$(X \cong Y \text{ in } pro^J\text{-C}) \Rightarrow (X \cong Y \text{ in } pro^K\text{-C})$$

holds because there exists the functor  $\underline{T} : pro^J\text{-C} \rightarrow pro^K\text{-C}$ . Conversely, let  $X \cong Y$  of  $pro^K\text{-C}$ . Choose any isomorphism  $g : X \rightarrow Y$  of  $pro^K\text{-C}$ , and let  $(g, g_\mu^k) : X \rightarrow Y$  of  $inv^K\text{-C}$  be any representative of  $g$ . Let us define

$$f = g : M \rightarrow \Lambda$$

and

$$(\forall \mu \in M)(\forall j \in J) f_\mu^j = g_\mu^{\phi(j)} : X_{f(\mu)} \rightarrow Y_\mu.$$

Since  $\phi$  is cofinal (i.e., for every  $k \in K$  there exists a  $j \in J$  such that  $\phi(j) \geq k$ ) and increasing (especially, for every  $j' \geq j$ ,  $\phi(j') \equiv k' \geq \phi(j) \geq k$ ) one can easily verify that

$$(f, f_\mu^j) : \mathbf{X} \rightarrow \mathbf{Y}$$

is a morphism of  $inv^J\text{-}\mathcal{C}$ , and thus, the equivalence class

$$\mathbf{f} = [(f, f_\mu^j)] : \mathbf{X} \rightarrow \mathbf{Y}$$

is a morphism of  $pro^J\text{-}\mathcal{C}$ . Let  $\mathbf{v} \equiv \mathbf{g}^{-1} : \mathbf{Y} \rightarrow \mathbf{X}$  of  $pro^K\text{-}\mathcal{C}$  be the inverse of  $\mathbf{g}$ , and let  $(v, v_\lambda^k) : \mathbf{Y} \rightarrow \mathbf{X}$  of  $inv^K\text{-}\mathcal{C}$  be any representative of  $\mathbf{v}$ . Let us define

$$u = v : \Lambda \rightarrow M$$

and

$$(\forall \lambda \in \Lambda)(\forall j \in J) u_\lambda^j = v_\lambda^{\phi(j)} : Y_{u(\lambda)} \rightarrow X_\lambda.$$

Now, as for  $(f, f_\mu^j)$  before, one readily verifies that

$$(u, u_\lambda^j) : \mathbf{Y} \rightarrow \mathbf{X}$$

is a morphism of  $inv^J\text{-}\mathcal{C}$ , and thus, the equivalence class

$$\mathbf{u} = [(u, u_\lambda^j)] : \mathbf{Y} \rightarrow \mathbf{X}$$

is a morphism of  $pro^J\text{-}\mathcal{C}$ . Since  $\mathbf{vg} = 1_{\mathbf{X}}$  and  $\mathbf{gv} = 1_{\mathbf{Y}}$  in  $pro^K\text{-}\mathcal{C}$ , the relations

$$(gv, v_\lambda^k g_{v(\lambda)}^k) \sim (1_\Lambda, 1_\lambda^k) : \mathbf{X} \rightarrow \mathbf{X}$$

and

$$(vg, g_\mu^k v_{g(\mu)}^k) \sim (1_M, 1_\mu^k) : \mathbf{Y} \rightarrow \mathbf{Y}$$

hold in  $inv^K\text{-}\mathcal{C}$ . Then, by our construction, one straightforwardly verifies that

$$(fu, u_\lambda^j f_{u(\lambda)}^j) \sim (1_\Lambda, 1_\lambda^j) : \mathbf{X} \rightarrow \mathbf{X}$$

and

$$(uf, f_{\mu}^j u_{f(\mu)}^j) \sim (1_M, 1_{\mu}^j) : Y \rightarrow Y$$

hold in  $inv^J\mathcal{C}$ . Therefore,  $u \equiv f^{-1}$  is the inverse of  $f$  in  $pro^J\mathcal{C}$ , implying that  $X \cong Y$  in  $pro^J\mathcal{C}$ .  $\square$

#### 4. The $J$ -shape category of a category

An enriched pro-category  $pro^J\mathcal{C}$  is interesting and useful by itself because, in general, it divides (classifies) the objects into larger classes (isomorphisms types) than the underlying pro-category  $pro\mathcal{C}$  (see Examples 7.1 and 7.2 of [26]). Moreover, in many important cases one can go on much further, i.e., to develop the corresponding  $J$ -shape theory.

Let  $\mathcal{D}$  be a full (not essential, but a convenient condition) and pro-reflective subcategory of  $\mathcal{C}$ . Let  $p : X \rightarrow X$  and  $p' : X \rightarrow X'$  be  $\mathcal{D}$ -expansions of the same object  $X$  of  $\mathcal{C}$ , and let  $q : Y \rightarrow Y$  and  $q' : X \rightarrow X'$  be  $\mathcal{D}$ -expansions of the same object  $Y$  of  $\mathcal{C}$ . Then there exist two canonical isomorphisms  $i : X \rightarrow X'$  and  $j : Y \rightarrow Y'$  of  $pro\mathcal{D}$ . Consequently, for every directed partially ordered set  $J$ , the (induced) morphisms  $\underline{i} \equiv \underline{I}(i) : X \rightarrow X'$  and  $\underline{j} \equiv \underline{I}(j) : Y \rightarrow Y'$  are isomorphisms of  $pro^J\mathcal{D}$ . A  $J$ -morphism  $f : X \rightarrow Y$  is said to be  $pro^J\mathcal{D}$  equivalent to a morphism  $f' : X' \rightarrow Y'$ , denoted by  $f \sim f'$ , if the following diagram in  $pro^J\mathcal{D}$  commutes:

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{j} & Y' \end{array}$$

According to the analogous facts in  $pro\mathcal{D}$ , and since  $\underline{I}$  is a functor, the diagram defines an equivalence relation on the appropriate subclass of  $Mor(pro^J\mathcal{D})$ , such that  $f \sim f'$  and  $g \sim g'$  imply  $gf \sim g'f'$  whenever these compositions exist. The equivalence class of such an  $f$  is denoted by  $\langle f \rangle$ . Further, given  $p, p', q, q'$  and  $f$ , there exists a unique  $f' (= j\underline{f}^{-1})$  such that  $f \sim f'$ .

We are now to define the (abstract)  $J$ -shape category  $Sh_{(\mathcal{C}, \mathcal{D})}^J$  for  $(\mathcal{C}, \mathcal{D})$  as follows. The objects of  $Sh_{(\mathcal{C}, \mathcal{D})}^J$  are all the objects of  $\mathcal{C}$ . A morphism  $F \in Sh_{(\mathcal{C}, \mathcal{D})}^J(X, Y)$  is the  $(pro^J\mathcal{D})$ -equivalence class  $\langle f \rangle$  of a  $J$ -morphism  $f : X \rightarrow Y$  of  $pro^J\mathcal{D}$ , with respect to any choice of a pair of  $\mathcal{D}$ -expansions  $p : X \rightarrow X, q : Y \rightarrow Y$ . In other words, a  $J$ -shape morphism  $F : X \rightarrow Y$  is given by a diagram

$$\begin{array}{ccc} X & \xleftarrow{p} & X \\ f \downarrow & & \downarrow F \\ Y & \xleftarrow{q} & Y \end{array}$$

in  $pro^J\mathcal{C}$ . The composition of such an  $F : X \rightarrow Y, F = \langle f \rangle$  and a  $G : Y \rightarrow Z, G = \langle g \rangle$ , is defined by the representatives, i.e.,  $GF : X \rightarrow Z, GF = \langle gf \rangle$ . The identity  $J$ -shape morphism on an object  $X, 1_X : X \rightarrow X$ , is the  $(pro^J\mathcal{D})$ -equivalence class  $\langle 1_X \rangle$  of the identity morphism  $1_X$  of  $pro^J\mathcal{D}$ . Since

$$Sh_{(\mathcal{C}, \mathcal{D})}^J(X, Y) \approx pro^J\mathcal{D}(X, Y)$$

is a set, the  $J$ -shape category  $Sh_{(\mathcal{C}, \mathcal{D})}^J$  is well defined. One may say that  $pro^J\mathcal{D}$  is the realizing category for the  $J$ -shape

category  $Sh_{(\mathcal{C}, \mathcal{D})}^J$ .

For every  $f : X \rightarrow Y$  of  $\mathcal{C}$  and every pair of  $\mathcal{D}$ -expansions  $p : X \rightarrow X, q : Y \rightarrow Y$ , there exists an  $f : X \rightarrow Y$  of  $pro^J\text{-}\mathcal{D}$ , such that the following diagram in  $pro^J\text{-}\mathcal{C}$  commutes:

$$\begin{array}{ccc} X & \xleftarrow{p} & X \\ f \downarrow & & \downarrow f \\ Y & \xleftarrow{q} & Y \end{array}$$

(Hereby, we consider  $\mathcal{C} \subseteq pro\text{-}\mathcal{C}$  to be subcategories of  $pro^J\text{-}\mathcal{C}$ !) The same  $f$  and another pair of  $\mathcal{D}$ -expansions  $p' : X \rightarrow X', q' : Y \rightarrow Y'$  yield an  $f' : X' \rightarrow Y'$  of  $pro^J\text{-}\mathcal{D}$ . Then, however,  $f \sim f'$  in  $pro^J\text{-}\mathcal{D}$  must hold. Thus, every morphism  $f \in \mathcal{C}(X, Y)$  yields a ( $pro^J\text{-}\mathcal{D}$ )-equivalence class  $\langle f \rangle$ , i.e. a  $J$ -shape morphism  $F \in Sh_{(\mathcal{C}, \mathcal{D})}^J(X, Y)$ . If one defines  $S^J(X) = X, X \in Ob\mathcal{C}$ , and  $S^J(f) = F = \langle f \rangle, f \in Mor\mathcal{C}$ , then

$$S^J \equiv S_{(\mathcal{C}, \mathcal{D})}^J : \mathcal{C} \rightarrow Sh_{(\mathcal{C}, \mathcal{D})}^J$$

becomes a functor, called the (abstract)  $J$ -shape functor. Comparing to the (abstract) shape functor, we know that the restriction of  $S^J$  to  $\mathcal{D}$  into the full subcategory of  $Sh_{(\mathcal{C}, \mathcal{D})}^J$ , determined by  $Ob\mathcal{D}$ , is not a category isomorphism (Example 3 of [26]). Nevertheless, we shall prove that  $P$  and  $Q$  are isomorphic objects of  $\mathcal{D}$  if and only if they are isomorphic in  $Sh_{(\mathcal{C}, \mathcal{D})}^J$ , i.e. they are of the same  $J$ -shape (Theorem 5 below). Thus, clearly, the  $J$ -shape type classification on  $\mathcal{D}$  coincides with the shape type classification. Further, recall that for every  $X \in Ob\mathcal{C}$  and every  $Q \in Ob\mathcal{D}$  the shape functor induces a bijection

$$S|\cdot : \mathcal{C}(X, Q) \rightarrow Sh_{(\mathcal{C}, \mathcal{D})}(X, Q).$$

However, in the same circumstances, the  $J$ -shape functor induces an injection

$$S^J|\cdot : \mathcal{C}(X, Q) \rightarrow Sh_{(\mathcal{C}, \mathcal{D})}^J(X, Q),$$

which, in general, is not a surjection (Example 3 of [26]). Finally, the functor  $Sh_{(\mathcal{C}, \mathcal{D})}^J$  factorizes as  $Sh_{(\mathcal{C}, \mathcal{D})}^J = I_{(\mathcal{C}, \mathcal{D})} S_{(\mathcal{C}, \mathcal{D})}$ , where  $S_{(\mathcal{C}, \mathcal{D})} : \mathcal{C} \rightarrow Sh_{(\mathcal{C}, \mathcal{D})}$  is the (abstract) shape functor, while  $I_{(\mathcal{C}, \mathcal{D})} : Sh_{(\mathcal{C}, \mathcal{D})} \rightarrow Sh_{(\mathcal{C}, \mathcal{D})}^J$  is induced by the “inclusion” functor  $I \equiv I_{\mathcal{D}} : pro\text{-}\mathcal{D} \rightarrow pro^J\text{-}\mathcal{D}$ . (This implies that the induced function  $\mathcal{C}(X, Q) \rightarrow Sh_{(\mathcal{C}, \mathcal{D})}^J(X, Q)$  is an injection.)

As in the case of the abstract shape, the most interesting example of the above construction is  $\mathcal{C} = HTop$ -the homotopy category of topological spaces and  $\mathcal{D} = HPol$ -the homotopy category of polyhedra (or  $\mathcal{D} = HANR$ -the homotopy category of ANR’s for metric spaces). In this case, one speaks about the (ordinary or standard)  $J$ -shape category

$$Sh_{(HTop, HPol)}^J \equiv Sh^J(Top) \equiv Sh^J$$

of topological spaces and of (ordinary or standard)  $J$ -shape functor

$$S^J : HTop \rightarrow Sh^J,$$

which factorizes as  $S^J = IS$ , where  $S : HTop \rightarrow Sh$  is the shape functor, and  $I : Sh \rightarrow Sh^J$  is induced by the “inclusion” functor  $\underline{I}_{HPol} \equiv pro\text{-}HPol \rightarrow pro^J\text{-}HPol$ . The realizing category for  $Sh^J$  is the category  $pro^J\text{-}HPol$  (or  $pro^J\text{-}HANR$ ). The underlying theory might be called the (ordinary or standard)  $J$ -shape theory (for topological spaces). Clearly, on locally nice spaces (polyhedra, CW-complexes, ANR’s, ...) the  $J$ -shape type classification coincides with the shape type classification and, consequently, with the homotopy type classification.

Similarly to the case of the shape of compacta, let us consider the homotopy (sub)category of compact metric spaces,  $HcM \subseteq HTop$ . Since  $HcPol \subseteq HcM$  and  $HcANR \subseteq HcM$  are “sequentially” pro-reflective (and homotopically equivalent) subcategories, there exist the  $J$ -shape category of compacta,

$$Sh^J(cM) \equiv Sh^J_{(HcM, HcPol)} \cong Sh^J_{(HcM, HcANR)},$$

and the corresponding (restriction of the)  $J$ -shape functor

$$S^J : HcM \rightarrow Sh^J(cM),$$

such that  $S^J = IS$ , where  $S : HcM \rightarrow Sh(cM)$  is the shape functor on compacta, and  $I : Sh(cM) \rightarrow Sh^J(cM)$  is induced by the “inclusion” functor  $\underline{I} : tow\text{-}HcPol \rightarrow tow^J\text{-}HcPol$  (or  $\underline{I} : tow\text{-}HcANR \rightarrow tow^J\text{-}HcANR$ ). The category  $Sh^J(cM)$  is a full subcategory of  $Sh^J$ . Notice that the realizing category for  $Sh^J(cM)$  is the category  $tow^J\text{-}HcPol$  as well as the category  $tow^J\text{-}HcANR$ .

The following facts are immediate consequences of Theorems 3 and 4 of the previous section.

**Corollary 2.** Let  $\mathcal{C}$  be a category and let  $\mathcal{D} \subseteq \mathcal{C}$  be a pro-reflective subcategory. Then

(i)  $Sh_{(\mathcal{C}, \mathcal{D})} = Sh_{(\mathcal{C}, \mathcal{D})}^{\{1\}}$ ;

(ii)  $Sh_{(\mathcal{C}, \mathcal{D})}^* = Sh_{(\mathcal{C}, \mathcal{D})}^{\mathbb{N}}$ ;

(iii) If  $J$  is a directed partially ordered set having  $\max J$ , then  $Sh_{(\mathcal{C}, \mathcal{D})}^J \cong Sh_{(\mathcal{C}, \mathcal{D})}$ .

**Corollary 3.** Let  $\mathcal{C}$  be a category, let  $\mathcal{D} \subseteq \mathcal{C}$  be a pro-reflective subcategory, let  $J$  be a well ordered set and let  $K$  be a directed partially ordered set, both without maximal elements. If there exists an increasing function  $\phi : J \rightarrow K$  such that  $\phi[J]$  is cofinal in  $K$ , then there exists a functor

$$T : Sh_{(\mathcal{C}, \mathcal{D})}^J \rightarrow Sh_{(\mathcal{C}, \mathcal{D})}^K$$

keeping the objects fixed, and  $T$  does not depend on  $\phi$ . Furthermore, for every pair  $X, Y$  of objects of  $\mathcal{C}$ , the equivalence

$$(X \cong Y \text{ in } Sh_{(\mathcal{C}, \mathcal{D})}^J) \Leftrightarrow (X \cong Y \text{ in } Sh_{(\mathcal{C}, \mathcal{D})}^K)$$

holds true.

An important property of a shape theory is that the shape type of a “nice” object of  $\mathcal{C}$  and its isomorphism class (in  $\mathcal{C}$ ) coincide. We are to show this property holds for a  $J$ -shape theory as well. Let  $\mathcal{D}$  be a full and pro-reflective subcategory of  $\mathcal{C}$ , let  $X \in Ob\mathcal{C}$  and let  $\mathbf{p} = (p_\lambda) : X \rightarrow \mathbf{X} = (X_\lambda, p_{\lambda\lambda}, \Lambda)$  be a  $\mathcal{D}$ -expansion of  $X$ . Further, let  $J$  be a directed partially ordered set, let  $Q \in Ob\mathcal{D}$  and let a family  $(\varphi^j)$  of  $\mathcal{C}$ -morphisms  $\varphi^j : X \rightarrow Q, j \in J$ , be given. We say that  $(\varphi^j)$  uniformly factorizes through  $\mathbf{p}$  if there exists a (fixed)  $\lambda \in \Lambda$  such that, for every  $j$ ,  $\varphi^j$  factorizes through  $p_\lambda$ . Such a family  $(\varphi^j)$  determines a  $J$ -shape morphism  $F : X \rightarrow Q$ . Indeed, then there is a  $\lambda \in \Lambda$  such that, for every  $j \in J$ , there exists a morphism  $f^j : X_\lambda \rightarrow Q$  of  $\mathcal{D}$  ( $\mathcal{D} \subseteq \mathcal{C}$  is full) satisfying  $\varphi^j = f^j p_\lambda$ . Hence, the family  $(f^j)$  (with the index function  $\{1\} \rightarrow \Lambda, 1 \mapsto \lambda$ ) determines a unique morphism  $\mathbf{f} = [(f^j)] : \mathbf{X} \rightarrow \mathbf{Q} = (Q)$  of  $pro^J\text{-}\mathcal{D}$ . Since  $\mathbf{1} : Q \rightarrow Q$  is a  $\mathcal{D}$ -expansion of  $Q$ , the morphism  $\mathbf{f}$  determines a unique  $J$ -shape morphism  $F = \langle \mathbf{f} \rangle : X \rightarrow Q$  of  $Sh_{(\mathcal{C}, \mathcal{D})}^J$ . We say that such an  $F$  is induced by  $(\varphi^j)$ .



Notice that the above construction depends on the index  $\lambda$ . The converse reads as follows.

**Lemma 7.** Let  $X \in \text{Ob}\mathcal{C}$ , let  $\mathbf{p} = (p_\lambda) : X \rightarrow \mathbf{X} = (X_\lambda, p_{\lambda\lambda}, \Lambda)$  be a  $\mathcal{D}$ -expansion of  $X$  and let  $Q \in \text{Ob}\mathcal{D}$ . Then, for every directed partially ordered set  $J$ , every  $J$ -shape morphism  $F : X \rightarrow Q$  of  $\text{Sh}_{(\mathcal{C}, \mathcal{D})}^J$  is induced by a family of morphisms  $\varphi^j : X \rightarrow Q$  of  $\mathcal{C}$ ,  $j \in J$ , such that  $(\varphi^j)$  uniformly factorizes through  $\mathbf{p}$ .

**Proof.** Let  $F : X \rightarrow Q$  be a  $J$ -shape morphism of  $\text{Sh}_{(\mathcal{C}, \mathcal{D})}^J$ . For  $\mathcal{D}$ -expansions  $\mathbf{p} = (p_\lambda) : X \rightarrow \mathbf{X}$  and  $\mathbf{1} : Q \rightarrow \mathbf{Q} = (Q)$ , there exists a representative  $\mathbf{f} : \mathbf{X} \rightarrow (Q)$  of  $\text{pro}^J\text{-}\mathcal{D}$  of  $F$ . Consequently, there exist a  $\lambda \in \Lambda$  and a family  $(f^j)$  of  $\mathcal{D}$ -morphisms  $f^j : X_\lambda \rightarrow Q$ ,  $j \in J$ , which determines  $\mathbf{f}$ . Then, by putting  $\varphi^j = f^j p_\lambda$ ,  $j \in J$ , one obtains the desired inducing family  $(\varphi^j)$  for  $F$ .  $\square$

Let  $(\varphi^j)$  and  $(\varphi'^j)$  uniformly factorize through the same  $\mathcal{D}$ -expansion  $\mathbf{p} : X \rightarrow \mathbf{X}$  (via a  $\lambda$  and a  $\lambda'$  respectively). Then  $(\varphi^j)$  is said to be almost equal to  $(\varphi'^j)$ , if there exist a  $\lambda_0 \geq \lambda, \lambda'$  and a  $j_0 \in J$  such that

$$(\forall j \geq j_0) f^j p_{\lambda\lambda_0} = f'^j p_{\lambda'\lambda_0}.$$

Clearly, it is an equivalence relation. Further, since  $\mathbf{p}$  is a  $\mathcal{D}$ -expansion,  $(\varphi^j)$  and  $(\varphi'^j)$  are almost equal, if and only if there exists a  $j_0 \in J$  such that  $\varphi^j = \varphi'^j : X \rightarrow Q$ , for all  $j \geq j_0$ .

**Lemma 8.** Let  $(\varphi^j)$  and  $(\varphi'^j)$  (of  $X \in \text{Ob}\mathcal{C}$  to  $Q \in \text{Ob}\mathcal{D}$ ) uniformly factorize through the same  $\mathcal{D}$ -expansion  $\mathbf{p} : X \rightarrow \mathbf{X}$ , and let  $F : X \rightarrow Q$  and  $F' : X \rightarrow Q$  of  $\text{Sh}_{(\mathcal{C}, \mathcal{D})}^J$  be induced by  $(\varphi^j)$  and  $(\varphi'^j)$  respectively. Then  $F = F'$  if and only if  $(\varphi^j)$  and  $(\varphi'^j)$  are almost equal.

**Proof.** Let  $(\varphi^j)$  and  $(\varphi'^j)$  uniformly factorize through the same  $\mathbf{p} : X \rightarrow \mathbf{X}$ , i.e., let there exist  $\lambda, \lambda' \in \Lambda$  such that, for every  $j \in J$ ,  $\varphi^j = f^j p_\lambda$  and  $\varphi'^j = f'^j p_{\lambda'}$ , where  $f^j : X_\lambda \rightarrow Q$  and  $f'^j : X_{\lambda'} \rightarrow Q$  are morphisms of  $\mathcal{D}$ . Let  $F : X \rightarrow Q$  and  $F' : X \rightarrow Q$  be the  $J$ -shape morphisms of  $\text{Sh}_{(\mathcal{C}, \mathcal{D})}^J$  induced by  $(\varphi^j)$  and  $(\varphi'^j)$  respectively. Let  $\mathbf{f}, \mathbf{f}' : \mathbf{X} \rightarrow (Q)$  of  $\text{pro}^J\text{-}\mathcal{D}$  be representatives of  $F, F'$  respectively. Now, if  $F = F'$  then  $\mathbf{f} = \mathbf{f}'$ , and  $\mathbf{f}, \mathbf{f}'$  are determined by the families  $(f^j), (f'^j)$  respectively. Therefore, there exist a  $\lambda_0 \geq \lambda, \lambda'$  and a  $j_0 \in J$  such that

$$(\forall j \geq j_0) f^j p_{\lambda\lambda_0} = f'^j p_{\lambda'\lambda_0}.$$

This means that  $(\varphi^j)$  and  $(\varphi'^j)$  are almost equal. Conversely, if  $(\varphi^j)$  and  $(\varphi'^j)$  are almost equal, then the corresponding families  $(f^j)$  and  $(f'^j)$  induce the same morphism  $\mathbf{f} : \mathbf{X} \rightarrow (Q)$  of  $\text{pro}^J\text{-}\mathcal{D}$ . Consequently, the families  $(\varphi^j)$  and  $(\varphi'^j)$  induce the same  $J$ -shape morphism  $F = \langle \mathbf{f} \rangle = F' : X \rightarrow Q$  of  $\text{Sh}_{(\mathcal{C}, \mathcal{D})}^J$ .  $\square$

Consider now the more special case where  $X \equiv P \in \text{Ob}\mathcal{D}$  too. Then  $\mathbf{1} : P \rightarrow \mathbf{P} = (P)$  and  $\mathbf{1} : Q \rightarrow \mathbf{Q} = (Q)$  are (the rudimentary)  $\mathcal{D}$ -expansions. Thus, every  $J$ -shape morphism  $F : P \rightarrow Q$  of  $\text{Sh}_{(\mathcal{C}, \mathcal{D})}^J$  is induced by a family of  $\mathcal{D}$ -morphisms  $f^j : P \rightarrow Q$ ,  $j \in J$ . Furthermore, any two such families  $(f^j)$  and  $(f'^j)$  induce the same  $J$ -shape morphism, if and only if  $f^j = f'^j$  for almost all  $j$  (all  $j \geq j_0$ , where  $j_0$  is a fixed index). This implies that there is a surjection

$$(\mathcal{D}(P, Q))^J \rightarrow \text{Sh}_{(\mathcal{C}, \mathcal{D})}^J(P, Q)$$

of the set of all  $J$ -families  $\Phi = (f^j)_{j \in J}$  of  $\mathcal{D}$ -morphisms  $f^j : P \rightarrow Q$  onto the set of all  $J$ -shape morphisms  $F : P \rightarrow Q$  of  $\text{Sh}_{(\mathcal{C}, \mathcal{D})}^J$ . Finally, one can readily see that if an  $F : P \rightarrow Q$  is induced by an  $(f^j)$  and a  $G : Q \rightarrow R$  is induced by a  $(g^j)$ , then the composition  $GF : P \rightarrow R$  is induced by  $(g^j f^j)$ . The following theorem generalizes Claim 3 of [26].

**Theorem 5.** Let  $\mathcal{D}$  be a pro-reflective subcategory of  $\mathcal{C}$  and let  $J$  be a directed partially ordered set. Then, for every pair  $P, Q \in \text{Ob}\mathcal{D}$ , the following statements are equivalent:

- (i)  $P$  and  $Q$  are isomorphic objects of  $\mathcal{D}$ ,  $P \cong Q$  in  $\mathcal{D} \subseteq \mathcal{C}$ ;
- (ii)  $P$  and  $Q$  have the same shape,  $Sh(P) = Sh(Q)$ , i.e.,  $P \cong Q$  in  $Sh_{(\mathcal{C}, \mathcal{D})}$ ;
- (iii)  $P$  and  $Q$  have the same  $J$ -shape,  $Sh^J(P) = Sh^J(Q)$ , i.e.,  $P \cong Q$  in  $Sh^J_{(\mathcal{C}, \mathcal{D})}$ .

**Proof.** The equivalence (i)  $\Leftrightarrow$  (ii) is the well known fact. The implication (ii)  $\Rightarrow$  (iii) follows by the functor  $I_{(\mathcal{C}, \mathcal{D})} : Sh_{(\mathcal{C}, \mathcal{D})} \rightarrow Sh^J_{(\mathcal{C}, \mathcal{D})}$ . Let  $P, Q \in Ob\mathcal{D}$  have the same  $J$ -shape. Then there exists a pair of  $J$ -shape isomorphisms  $F : P \rightarrow Q$ ,  $G : Q \rightarrow P$  such that  $GF = 1_P$  and  $FG = 1_Q$  in  $Sh^J_{(\mathcal{C}, \mathcal{D})}$ . By the above consideration, there exist families  $(f^j)$  and  $(g^j)$  of  $\mathcal{D}$ -morphisms  $f^j : P \rightarrow Q$  and  $g^j : Q \rightarrow P, j \in J$ , which induce  $F$  and  $G$  respectively. Furthermore, the families  $(g^j f^j)$  and  $(f^j g^j)$  induce  $1_P$  and  $1_Q$  (of  $Sh^J_{(\mathcal{C}, \mathcal{D})}$ ). Since the constant family  $(1^j_P = 1_P)$  and  $(1^j_Q = 1_Q)$  also induce  $1_P$  and  $1_Q$  (of  $Sh^J_{(\mathcal{C}, \mathcal{D})}$ ) respectively, Lemma 8 implies that  $g^j f^j = 1_P$  and  $f^j g^j = 1_Q$  hold for almost all  $j \in J$ . Consequently,  $P$  and  $Q$  are isomorphic objects of  $\mathcal{D}$ , and thus, (iii)  $\Rightarrow$  (i).  $\square$

## 5. The continuity theorem for $J$ -shape

A very important benefit of the standard shape theory comparing to the homotopy theory is the continuity property, i.e., the category  $Sh$  admits the limit functor, while it fails for  $HTop$ . Moreover, in general, every (abstract) shape theory has the continuity property (Theorem I.2.6. of [13]). Further, the continuity property holds for every coarse and every weak shape theory (Theorems 1 and 2 of [24]). We shall prove hereby that every  $J$ -shape theory has the continuity property as well.

**Theorem 6.** Let  $\mathcal{D}$  be a pro-reflective subcategory of  $\mathcal{C}$  and let  $J$  be a directed partially ordered set. Let  $X, Y \in Ob\mathcal{C}$ , let  $\mathbf{q} = (q_\mu) : Y \rightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$  be a  $\mathcal{C}$ -expansion of  $Y$  and let  $\mathbf{H} = (H_\mu) : X \rightarrow S^J(\mathbf{Y})$  be a morphism of  $pro-Sh^J_{(\mathcal{C}, \mathcal{D})}$ . Then there exists a unique  $J$ -shape morphism  $F : X \rightarrow Y$  such that  $\mathbf{H} = \mathbf{Q}F$ , where  $\mathbf{Q} = (Q_\mu) = S^J(\mathbf{q}) : Y \rightarrow S^J(\mathbf{Y})$  is the morphism of  $pro-Sh^J_{(\mathcal{C}, \mathcal{D})}$  induced by  $\mathbf{q}$ , i.e., for every  $\mu \in M$ ,  $H_\mu = Q_\mu F$ , and  $Q_\mu$  is induced by  $q_\mu$ ,  $Q_\mu = S^J(q_\mu)$ . In other words, if  $\mathbf{q} : Y \rightarrow \mathbf{Y}$  is a  $\mathcal{C}$ -expansion, then  $\mathbf{Q} = S^J(\mathbf{q}) : Y \rightarrow S^J(\mathbf{Y})$  is an inverse limit in  $Sh^J_{(\mathcal{C}, \mathcal{D})}$ , i.e., every  $\mathcal{C}$ -expansion  $\mathbf{q} : Y \rightarrow \mathbf{Y}$  induces, for each  $X$ , a bijection

$$pro-Sh^J_{(\mathcal{C}, \mathcal{D})}(\lfloor X \rfloor, S^J(\mathbf{Y})) \approx Sh^J_{(\mathcal{C}, \mathcal{D})}(X, Y),$$

defined by the following diagram

$$\begin{array}{ccc} & X & \\ & \downarrow F & \\ \mathbf{H} & \swarrow & \\ S^J(\mathbf{Y}) & \xleftarrow{S^J(\mathbf{q})} & Y \end{array}$$

The proof consists of two steps. In the first one we consider the special case of a  $\mathcal{D}$ -expansion  $\mathbf{q} : Y \rightarrow \mathbf{Y}$ .

**Lemma 9.** Let  $\mathcal{D}$  be a pro-reflective subcategory of  $\mathcal{C}$  and let  $J$  be a directed partially ordered set. Let  $X, Y \in Ob\mathcal{C}$ , let  $\mathbf{q} = (q_\mu) : Y \rightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$  be a  $\mathcal{D}$ -expansion of  $Y$  and let  $\mathbf{H} = (H_\mu) : X \rightarrow S^J(\mathbf{Y})$  be a morphism of  $pro-Sh^J_{(\mathcal{C}, \mathcal{D})}$ . Then there exists a unique morphism  $F : X \rightarrow Y$  of  $Sh^J_{(\mathcal{C}, \mathcal{D})}$  for every  $\mu \in M$ ,  $H_\mu = S^J(q_\mu)F$ .

**Proof.** Let  $X, Y \in Ob\mathcal{C}$ , let  $\mathbf{q} = (q_\mu) : Y \rightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$  be a  $\mathcal{D}$ -expansion of  $Y$ . Let  $\mathbf{H} = (H_\mu) : X \rightarrow S^J(\mathbf{Y})$  be a morphism of  $pro-Sh^J_{(\mathcal{C}, \mathcal{D})}$  such that, for every related pair  $\mu \leq \mu', H_\mu = S^J(q_{\mu\mu'})H_{\mu'}$ . Let  $\mathbf{p} = (p_\lambda) : X \rightarrow \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$  be a  $\mathcal{D}$ -expansion of  $X$ . Since every  $Y_\mu \in Ob\mathcal{D}$ , every  $J$ -shape morphism  $H_\mu$  is represented by a unique morphism  $\mathbf{f}_\mu = [(f_\mu^j)] : X \rightarrow \lfloor Y_\mu \rfloor$  of  $pro^J\mathcal{D}(X, \lfloor Y_\mu \rfloor)$  ( $\lfloor Y_\mu \rfloor$  is the rudimentary system associated with  $Y_\mu$ ), via the following diagram

$$\begin{array}{ccc} \mathbf{X} & \xleftarrow{p} & X \\ \mathbf{f}_\mu \downarrow & & \downarrow H_\mu, \\ [Y_\mu] & \xleftarrow{1_{Y_\mu}} & Y_\mu \end{array}$$

where  $f_\mu^j : X_{\lambda(\mu)} \rightarrow Y_\mu, j \in J$ , are morphisms of  $\mathcal{D}$ . Further, since  $H_\mu = S^J(q_{\mu\mu'})H_{\mu'}$ ,  $\mu \leq \mu'$ , and  $S^J(q_{\mu\mu'})$  is represented by  $q_{\mu\mu'} = [(q_{\mu\mu'}^j = q_{\mu\mu'}^j)]$ , i.e.,

$$\begin{array}{ccc} [Y_\mu] & \xleftarrow{1_{Y_\mu}} & Y_\mu \\ \mathbf{q}_{\mu\mu'} \uparrow & & \uparrow S^J(q_{\mu\mu'}), \\ [Y_{\mu'}] & \xleftarrow{1_{Y_{\mu'}}} & Y_{\mu'} \end{array}$$

it follows that  $\mathbf{f}_\mu = \mathbf{q}_{\mu\mu'}\mathbf{f}_{\mu'}$  in  $pro^J\mathcal{D}$ ,  $\mu \leq \mu'$ . Thus, the following diagram in  $pro^J\mathcal{D}$  commutes

$$\begin{array}{ccc} & \mathbf{X} & \\ \cdots \mathbf{f}_\mu \swarrow & & \searrow \mathbf{f}_{\mu'} \cdots \\ \cdots \leftarrow [Y_\mu] & \xleftarrow{\mathbf{q}_{\mu\mu'}} & [Y_{\mu'}] \leftarrow \cdots \end{array}$$

This means that, for every pair  $\mu \leq \mu'$ , there exist a  $\lambda \geq \lambda(\mu), \lambda(\mu')$  and a  $j \in J$  such that, for every  $j' \geq j$ , the following diagram in  $\mathcal{D}$  commutes:

$$\begin{array}{ccc} & X_\lambda & \\ & \swarrow p_{\lambda(\mu)\lambda} & \swarrow p_{\lambda(\mu')\lambda} \\ X_{\lambda(\mu)} & & X_{\lambda(\mu')} \\ \mathbf{f}_\mu^{j'} \downarrow & & \downarrow \mathbf{f}_{\mu'}^{j'} \\ Y_\mu & \xleftarrow{q_{\mu\mu'}} & Y_{\mu'} \end{array}$$

Let us define a function  $f : M \rightarrow \Lambda$  by putting  $f(\mu) = \lambda(\mu)$ . Then the ordered pair  $(f, (f_\mu^j)_{\mu \in M, j \in J})$  determines a  $J$ -morphism  $(f, f_\mu^j)$  of  $X$  to  $Y$  of  $inv^J\mathcal{D}$ . Thus, the class  $\mathbf{f} = [(f, f_\mu^j)] : X \rightarrow Y$  is a morphism of  $pro^J\mathcal{D}$ . Since  $\mathbf{p} : X \rightarrow X$  and  $\mathbf{q} : Y \rightarrow Y$  are  $\mathcal{D}$ -expansions, the diagram

$$\begin{array}{ccc} \mathbf{X} & \xleftarrow{p} & X \\ \mathbf{f} \downarrow & & \downarrow F \\ \mathbf{Y} & \xleftarrow{q} & Y \end{array}$$

represents a  $J$ -shape morphism  $F : X \rightarrow Y$ . Notice that, by construction,

$$S^J(q_\mu)F = H_\mu$$

holds for every  $\mu \in M$ . Moreover, such an  $F$  is unique because  $\mathbf{q} : Y \rightarrow \mathbf{Y}$  is a  $\mathcal{D}$ -expansion, and thus,  $(\text{pro-}S^J)(\mathbf{Y})$  and  $S^J(q_\mu)$  may be considered to be  $\mathbf{Y}$  and  $q_\mu, \mu \in M$ , respectively. Therefore, for every  $X$ , the correspondence  $\mathbf{H} = (H_\mu) \mapsto F$ , induced by  $\mathbf{q}$ , defines a bijection of  $\text{pro-}Sh_{(\mathcal{C}, \mathcal{D})}^J([X], \mathbf{Y})$  onto  $Sh_{(\mathcal{C}, \mathcal{D})}^J(X, Y)$ .  $\square$

**Proof.** (of Theorem 6) Let  $\mathbf{q} = (q_\mu) : Y \rightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$  be a  $\mathcal{C}$ -expansion of  $Y$ . Firstly, if  $F : X \rightarrow Y$  is a morphism of  $Sh_{(\mathcal{C}, \mathcal{D})}^J$ , then all  $F_\mu = S^J(q_\mu)F, \mu \in M$ , define a morphism of  $\mathbf{H} = (F_\mu) : [X] \rightarrow S^J(\mathbf{Y})$  of  $\text{pro-}Sh_{(\mathcal{C}, \mathcal{D})}^J$ , because  $F_\mu = S^J(q_{\mu\mu'})F_{\mu'}, \mu \leq \mu'$ . Conversely, let an  $\mathbf{H} = (H_\mu) \in \text{pro-}Sh_{(\mathcal{C}, \mathcal{D})}^J([X], S^J(\mathbf{Y}))$  be given. Choose any  $\mathcal{D}$ -expansion

$$\mathbf{q}' = (q'_v) : Y \rightarrow \mathbf{Y}' = (Y'_v, q'_{vv'}, N)$$

of  $Y$  ( $\mathcal{D}$  is a pro-reflective subcategory of  $\mathcal{C}$ !). Since  $\mathbf{q}$  is a  $\mathcal{C}$ -expansion (with respect to  $\mathcal{D}$ ), there exists a unique  $\mathbf{g} : Y \rightarrow \mathbf{Y}'$  of  $\text{pro-}\mathcal{C}$  such that  $\mathbf{g}\mathbf{q} = \mathbf{q}'$ . Let  $(g, g_v)$  be a representative of  $\mathbf{g}$  in  $\text{inv-}\mathcal{C}$ . For every  $v \in N$ , denote by  $G_v : Y_{g(v)} \rightarrow Y'_v$  the morphism of  $Sh_{(\mathcal{C}, \mathcal{D})}^J$  induced by  $g_v$ , i.e.,  $G_v = S^J(g_v)$ . Similarly, denote  $Q'_v = S^J(q'_v) : Y \rightarrow Y'_v$  and  $Q'_{vv'} = S^J(q'_{vv'}) : Y'_{v''} \rightarrow Y'_v, v \leq v'$ . Then, since  $(g, g_v)$  is a morphism of  $\text{inv-}\mathcal{C}$ , one readily sees that  $(g, G_v) : S^J(\mathbf{Y}) \rightarrow S^J(\mathbf{Y}')$  is a morphism of  $\text{inv-}Sh_{(\mathcal{C}, \mathcal{D})}^J$ . Thus, the equivalence class  $\mathbf{G} = [(g, G_v)] : S^J(\mathbf{Y}) \rightarrow S^J(\mathbf{Y}')$  is a morphism of  $\text{pro-}Sh_{(\mathcal{C}, \mathcal{D})}^J$ . Let  $\mathbf{F} = (F_v) : [X] \rightarrow S^J(\mathbf{Y}')$  of  $\text{pro-}Sh_{(\mathcal{C}, \mathcal{D})}^J$  be the composition of  $\mathbf{H}$  and  $\mathbf{G}$ . Then  $F_v = G_v H_{g(v)}, v \in N$ , and  $F_v = Q'_{vv'} F_{v'}, v \leq v'$ . By Lemma 9, there exists a unique  $F : X \rightarrow Y$  of  $Sh_{(\mathcal{C}, \mathcal{D})}^J$ , such that, for every  $v \in N, Q'_v F = F_v$ . This means that, for every  $v \in N$ ,

$$Q'_v F = G_v H_{g(v)}, \text{i.e.,}$$

$$S^J(q'_v)F = S^J(g_v)H_{g(v)}.$$

We have to prove that, for every  $\mu \in M, S^J(q_\mu)F = H_\mu$  holds. Firstly, we will prove the following statement:

$$(\forall \mu \in M)(\forall P \in \text{Ob}\mathcal{D})(\forall u \in \mathcal{C}(Y_\mu, P))$$

$$S^J(u)S^J(q_\mu)F = S^J(u)H_\mu.$$

Notice that a  $u : Y_\mu \rightarrow P$  of  $\mathcal{C}$  yields a unique  $\mathbf{u} = [(u)] : \mathbf{Y} \rightarrow [P]$  of  $\text{pro-}\mathcal{C}$ . Observe that  $\mathbf{u}\mathbf{q}$  is a (rudimentary) morphism  $uq_\mu = Y \rightarrow P$  belonging to  $\mathcal{C}$ . Since  $\mathbf{q}'$  is a  $\mathcal{D}$ -expansion and  $P \in \text{Ob}\mathcal{D}$ , there exists a unique  $\mathbf{v} : \mathbf{Y}' \rightarrow [P]$  of  $\text{pro-}\mathcal{D}$  (represented by a  $v_v : Y'_v \rightarrow P$  of  $\mathcal{D}$ ) such that  $\mathbf{v}\mathbf{q}' = \mathbf{u}\mathbf{q}$ . Then,  $\mathbf{u}\mathbf{q} = \mathbf{v}\mathbf{g}\mathbf{q}$ , which implies that  $\mathbf{u} = \mathbf{v}\mathbf{g}$ . This means that there exists a  $\mu' \geq \mu, g(v)$  such that

$$uq_{\mu'} = v g_v q_{g(v)\mu'}.$$

Now one calculates in a straightforward way that

$$\begin{aligned}
& S^J(u)S^J(q_\mu)F \\
&= S^J(u)S^J(q_{\mu'})S^J(q_{\mu'})F = S^J(uq_{\mu'})S^J(q_{\mu'})F = S^J(vg_vq_{g(v)\mu'})S^J(q_{\mu'})F \\
&= S^J(v)S^J(g_vq_{g(v)\mu'})S^J(q_{\mu'})F = S^J(v)S^J(g_vq_{g(v)})F = S^J(v)S^J(q'_v)F \\
&= S^J(v)S^J(g_v)H_{g(v)} = S^J(v)S^J(g_v)S^J(q_{g(v)\mu'})H_{\mu'} = S^J(vg_vq_{g(v)\mu'})H_{\mu'} \\
&= S^J(uq_{\mu'})H_{\mu'} = S^J(u)S^J(q_{\mu'})H_{\mu'} = S^J(u)H_\mu,
\end{aligned}$$

which proves the statement. Given a  $\mu \in M$ , let

$$\mathbf{q}^\mu = (q_\alpha^\mu) : Y_\mu \rightarrow \mathbf{Y}^\mu = (Y_\alpha^\mu, q_{\alpha\alpha}^\mu, A^\mu)$$

be a  $\mathcal{D}$ -expansion of  $Y_\mu$ . Then, by the above statement, for every  $\alpha \in A^\mu$ ,

$$S^J(q_\alpha^\mu)S^J(q_\mu)F = S^J(q_\alpha^\mu)H_\mu.$$

According to the definition of the  $J$ -shape category  $Sh_{(\mathcal{C}, \mathcal{D})}^J$ , this means that the  $J$ -shape morphisms

$$S^J(q_\mu)F, H_\mu : X \rightarrow Y_\mu$$

admit the same representing morphism  $f : X \rightarrow \mathbf{Y}^\mu$  of  $pro^J\text{-}\mathcal{D}$ . Thus,

$$S^J(q_\mu)F = H_\mu.$$

Finally, such an  $F$  is unique because

$$S^J(q_\mu)F = S^J(q_\mu)F', \mu \in M,$$

immediately implies

$$S^J(q'_v)F = S^J(q'_v)F', v \in N,$$

which means that  $F = F'$ . □

## 6. A $J$ -shape isomorphism

In this section, we are going to establish an analogue of the well known Morita lemma of [9], which should

characterize a  $J$ -shape isomorphism in an elegant and rather operative manner. According to the “reindexing theorem” (Theorem 2.) and definition of the abstract  $J$ -shape category  $Sh_{(\mathcal{C}, \mathcal{D})}^J$ , it suffices to characterize an isomorphism  $f \in \text{pro}^J\text{-}\mathcal{D}(X, Y)$  which admits a level representative  $(1_\Lambda, f_\lambda^j) : X \rightarrow Y$  of  $\text{inv}^J\text{-}\mathcal{D}$ . In the case of inverse sequences, a strictly increasing simple representative will do. Since the characterization does not depend on the objects of  $\mathcal{D}$ , we shall consider such an  $f$  of  $\text{pro}^J\text{-}\mathcal{C}$  as well as the special case of  $\text{tow}^{\mathbb{N}}\text{-}\mathcal{C}$ .

**Theorem 7.** Let  $\mathcal{C}$  be a category and let  $J$  be a directed partially ordered set. Let  $X = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$  and  $Y = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$  be inverse systems in  $\mathcal{C}$  over the same index set  $\Lambda$  and let a morphism  $f : X \rightarrow Y$  of  $\text{pro}^J\text{-}\mathcal{C}$  admit a level representative  $(1_\Lambda, f_\lambda^j)$ . Then  $f$  is an isomorphism if and only if, for every  $\lambda \in \Lambda$ , there exist  $\lambda' \geq \lambda$  and a  $j_\lambda \in J$  such that, for every  $j \geq j_\lambda$ , there exists a  $\mathcal{C}$ -morphism  $h_\lambda^j : Y_{\lambda'} \rightarrow X_\lambda$  so that the following diagram in  $\mathcal{C}$  commutes:

$$\begin{array}{ccc} X_\lambda & \leftarrow & X_{\lambda'} \\ f_\lambda^j \downarrow & h_\lambda^j \nearrow & \downarrow f_{\lambda'}^j \\ Y_\lambda & \leftarrow & Y_{\lambda'} \end{array}$$

**Proof.** Let  $f : X \rightarrow Y$  be an isomorphism of  $\text{pro}^J\text{-}\mathcal{C}$  which admits a level representative  $(1_\Lambda, f_\lambda^j)$ . Let  $f^{-1} \equiv g = [(g, g_\lambda^j)] : Y \rightarrow X$  be the inverse of  $f$ , i.e.

$$(g, g_\lambda^j)(1_\Lambda, f_\lambda^j) \sim (1_\Lambda, 1_{X_\lambda}) \wedge (1_\Lambda, f_\lambda^j)(g, g_\lambda^j) \sim (1_\Lambda, 1_{Y_\lambda}).$$

Given any  $\lambda \in \Lambda$ , choose  $\lambda'_1, \lambda'_2 \in \Lambda$  according to the above equivalence relations. Then there exists a  $\lambda' \geq \lambda'_1, \lambda'_2$ . Thus  $\lambda' \geq \lambda, g(\lambda)$ . Further, choose  $j_1, j_2 \in J$  according to the above equivalence relations and the given  $\lambda$ . Since  $(1_\Lambda, f_\lambda^j)$  is an  $J$ -morphism, for the pair  $g(\lambda) \leq \lambda'$ , there exists a  $j_3 \in J$  such that the appropriate commutativity condition holds. Since  $J$  is directed, there exists a  $j_\lambda \geq j_1, j_2, j_3$ . Let us define, for every  $j \geq j_\lambda$ , a morphism  $h_\lambda^j : Y_{\lambda'} \rightarrow X_\lambda$  of  $\mathcal{C}$  by putting

$$h_\lambda^j = g_\lambda^j q_{g(\lambda)\lambda'}.$$

We are to prove that the needed diagram commutes. Firstly, according to the second equivalence relation,

$$f_\lambda^j h_\lambda^j = f_\lambda^j g_\lambda^j q_{g(\lambda)\lambda'} = q_{\lambda\lambda'}.$$

Thus, the left (lower) triangle in the diagram commutes. Further, since  $j \geq j_3$ ,

$$h_\lambda^j f_{\lambda'}^j = g_\lambda^j q_{g(\lambda)\lambda'} f_{\lambda'}^j = g_\lambda^j f_{g(\lambda)\lambda'}^j p_{g(\lambda)\lambda'},$$

while, according to the first equivalence relation,

$$g_\lambda^j f_{g(\lambda)\lambda'}^j p_{g(\lambda)\lambda'} = p_{\lambda\lambda'}.$$

Therefore,

$$h_{\lambda}^j f_{\lambda}^j = p_{\lambda\lambda'},$$

which proves commutativity of the right (upper) triangle in the diagram.

Conversely, suppose that a morphism  $f = [(1_{\Lambda}, f_{\lambda}^j)] : X \rightarrow Y$  of  $pro^J\mathcal{C}$  fulfils the condition of the theorem. Let  $g : \Lambda \rightarrow \Lambda$  be defined by that condition, i.e., for each  $\lambda$ , choose and fix a  $g(\lambda) = \lambda' \geq \lambda$  by the condition. Further, for each  $\lambda \in \Lambda$  and every  $j \in J$ , a morphism  $g_{\lambda}^j : Y_{g(\lambda)} \rightarrow X_{\lambda}$  of  $\mathcal{C}$  by putting

$$g_{\lambda}^j = \begin{cases} h_{\lambda}^{j\lambda}; j \not\geq j_{\lambda} \\ h_{\lambda}^j; j \geq j_{\lambda} \end{cases},$$

where  $h_{\lambda}^j$  comes from the condition. We have to prove that  $(g, g_{\lambda}^j) : Y \rightarrow X$  is a  $J$ -morphism. Let a pair  $\lambda \leq \lambda'$  be given. Choose a  $\lambda_0 \geq g(\lambda)$ ,  $g(\lambda')$  and put  $\lambda_1 \geq g(\lambda_0)$ . Since  $(1_{\Lambda}, f_{\lambda}^j)$  is a  $J$ -morphism, for the pairs  $g(\lambda) \leq \lambda_0$  and  $g(\lambda') \leq \lambda_0$ , there exist  $j_1, j_2 \in J$  such that the appropriate commutativity conditions hold respectively. Since  $J$  is directed, there exists a

$$j \geq j_{\lambda}, j_{\lambda'}, j_{\lambda_0}, j_1, j_2.$$

Now, for every  $j' \geq j$ , consider the following corresponding diagram:

$$\begin{array}{ccccccc} X_{\lambda} & \leftarrow & X_{\lambda'} & \leftarrow & X_{g(\lambda')} & & \\ & \swarrow & & \swarrow & \downarrow & \swarrow & \\ & X_{g(\lambda)} & & & Y_{g(\lambda')} & & \\ & \downarrow & & & \downarrow & & \\ & Y_{g(\lambda)} & \leftarrow & & Y_{\lambda_0} & \leftarrow & Y_{\lambda_1} \end{array} \quad (1)$$

We shall prove, by chasing diagram (1), that

$$g_{\lambda}^{j'} q_{g(\lambda)\lambda_1} = p_{\lambda\lambda'} g_{\lambda}^{j'} q_{g(\lambda')\lambda_1}. \quad (2)$$

Since  $j' \geq j_{\lambda_0}$ , the condition of the theorem implies

$$g_{\lambda}^{j'} q_{g(\lambda)\lambda_1} = h_{\lambda}^{j'} q_{g(\lambda)\lambda_0} f_{\lambda_0}^{j'} h_{\lambda_0}^{j'}. \quad (3)$$

Since  $j' \geq j_1$ ,

$$h_{\lambda}^{j'} q_{g(\lambda)\lambda_0} f_{\lambda_0}^{j'} h_{\lambda_0}^{j'} = h_{\lambda}^{j'} f_{g(\lambda)\lambda_0}^{j'} p_{g(\lambda)\lambda_0} h_{\lambda_0}^{j'}. \quad (4)$$

Since  $j' \geq j_\lambda, j_\lambda$ , the condition of the theorem implies

$$h_\lambda^{j'} f_{g(\lambda)}^{j'} p_{g(\lambda)\lambda_0} h_{\lambda_0}^{j'} = p_{\lambda\lambda_0} h_{\lambda_0}^{j'} = p_{\lambda\lambda'} h_{\lambda'}^{j'} f_{g(\lambda')}^{j'} p_{g(\lambda')\lambda_0} h_{\lambda_0}^{j'}. \quad (5)$$

Since  $j' \geq j_2$ ,

$$p_{\lambda\lambda'} h_{\lambda'}^{j'} f_{g(\lambda')}^{j'} p_{g(\lambda')\lambda_0} h_{\lambda_0}^{j'} = p_{\lambda\lambda'} h_{\lambda'}^{j'} q_{g(\lambda')\lambda_0} f_{\lambda_0}^{j'} h_{\lambda_0}^{j'}. \quad (6)$$

Finally, since  $j' \geq j_{\lambda_0}$ , the condition of the theorem implies

$$p_{\lambda\lambda'} h_{\lambda'}^{j'} q_{g(\lambda')\lambda_0} f_{\lambda_0}^{j'} h_{\lambda_0}^{j'} = p_{\lambda\lambda'} h_{\lambda'}^{j'} q_{g(\lambda')g(\lambda_0)} = p_{\lambda\lambda'} g_{\lambda'}^{j'} q_{g(\lambda')\lambda_1}. \quad (7)$$

Now, by combining (3), (4), (5), (6) and (7), one establishes (2), which proves that  $(g, g_\lambda^j)$  is a  $J$ -morphism. Moreover, by the condition of the theorem, it is readily seen that, for each  $\lambda \in \Lambda$  and every  $j' \in J, j' \geq j_\lambda$ ,

$$g_\lambda^{j'} f_{g(\lambda)}^{j'} = h_\lambda^{j'} f_{g(\lambda)}^{j'} = p_{\lambda g(\lambda)} \wedge f_\lambda^{j'} g_\lambda^{j'} = f_\lambda^{j'} h_\lambda^{j'} = q_{\lambda g(\lambda)}.$$

This shows that

$$(g, g_\lambda^j)(1_\Lambda, f_\lambda^j) \sim (1_\Lambda, 1_{X_\lambda}) \wedge (1_\Lambda, f_\lambda^j)(g, g_\lambda^j) \sim (1_\Lambda, 1_{Y_\lambda}),$$

which means that  $\mathbf{g} = [(g, g_\lambda^j)] : \mathbf{Y} \rightarrow \mathbf{X}$  be the inverse of  $\mathbf{f}$ . Therefore,  $\mathbf{f}$  is an isomorphism of  $pro^J\text{-}\mathcal{C}$ .  $\square$

**Remark 3.** Since  $pro\text{-}\mathcal{C} = pro^{\{1\}}\text{-}\mathcal{C}$ , the original Morita lemma is the simplest case of Theorem 7. Further, since the coarse shape category  $Sh_{(\mathcal{C}, \mathcal{D})}^*$  is the  $\mathbb{N}$ -shape category  $Sh_{(\mathcal{C}, \mathcal{D})}^{\mathbb{N}}$ , Theorem 7 is a generalization of [26], Theorem 6.1.

One can easily verify that the condition (of Theorem 7) characterizing an isomorphism may be reduced to a cofinal subset  $\Lambda' \subseteq \Lambda$ . Thus, the following corollary holds.

**Corollary 4.** If an  $\mathbf{f} = [(1_\Lambda, f_\lambda^j)] : \mathbf{X} \rightarrow \mathbf{Y}$  of  $pro^J\text{-}\mathcal{C}$  admits a cofinal subset  $\Lambda' \subseteq \Lambda$  such that, for every  $\lambda' \in \Lambda'$ , there exists a  $j \in J$ , so that, for every  $j' \geq j$ ,  $f_{\lambda'}^{j'}$  is an isomorphism of  $\mathcal{C}$ , then  $\mathbf{f}$  is an isomorphism.

For the sake of completeness and unifying notations, we include hereby Theorem 6.4 of [26] (see also [23], Section 2) concerning the special case of inverse sequences and  $J = \mathbb{N}$ . It is very useful, for instance, in detecting an  $\mathbb{N}$ -shape (i.e., a coarse shape) isomorphism of metrizable compacta (i.e., in the case  $(\mathcal{C}, \mathcal{D}) = (HcM, HcPol)$ ).

**Theorem 8.** Let  $\mathbf{X} = (X_n, p_{nn})$  and  $\mathbf{Y} = (Y_m, q_{mm})$  be inverse sequences in a category  $\mathcal{C}$ , let  $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$  be a morphism of  $toW^{\mathbb{N}}\text{-}\mathcal{C}$  and let  $(f, f_m^j)$  be any simple representative of  $\mathbf{f}$  with a commutativity radius  $\gamma$  and  $f$  strictly increasing. If for every  $j \in \mathbb{N}$  and every  $m = 1, \dots, \gamma(j) - 1$ , there exists a  $\mathcal{C}$ -morphism  $h_{f(m)}^j : Y_{m+1} \rightarrow X_{f(m)}$  such that the diagram

$$\begin{array}{ccc} X_{f(m)} & \leftarrow & X_{f(m+1)} \\ f_m^j \downarrow & h_{f(m)}^j \swarrow & \downarrow f_{m+1}^j \\ Y_m & \leftarrow & Y_{m+1} \end{array}$$

in  $\mathcal{C}$  commutes, then  $\mathbf{f}$  is an isomorphism of  $toW^{\mathbb{N}}\text{-}\mathcal{C}$ .



Conversely, if  $f$  is an isomorphism of  $\text{tow}^{\mathbb{N}}\mathcal{C}$ , then, for every  $m \in \mathbb{N}$ , there exist an  $m' \geq m$  and a  $j \in \mathbb{N}$  such that, for every  $j' \geq j$ , there exists a  $\mathcal{C}$ -morphism  $h_{f(m)}^{j'} : Y_{m'} \rightarrow X_{f(m)}$  so that the following diagram

$$\begin{array}{ccc} X_{f(m)} & \leftarrow & X_{f(m')} \\ f_m^j \downarrow & h_{f(m)}^{j'} \swarrow & \downarrow f_m^{j'} \\ Y_m & \leftarrow & Y_{m'} \end{array}$$

in  $\mathcal{C}$  commutes.

**Remark 4.** We give no additional example but those of [21-23] and [26], though one can, by means of them, easily construct some with  $J = (\mathbb{N}, \leq')$  (for instance, in the case of  $m \leq' n$  iff  $\frac{n}{m} \in \mathbb{N}$ ). However, although the categories  $\text{pro}^* \text{-} H\text{Top}$  and  $\text{pro}^J \text{-} H\text{Top}$  (in this case) are pretty different, their isomorphism classifications coincide (Corollary 3). In order to find a new example, one should use an uncountable well ordered  $(J, \leq)$  and the transfinite induction technique. It leads to a complete new paper. Another possible way might be, firstly, to find a suitable invariant of the  $J$ -shape ( $\text{pro}^J \text{-} H\text{Top}$ ) and, secondly, to prove that it is not an invariant of the coarse shape ( $\text{pro}^* \text{-} H\text{Top}$ ), which again asks for a transfinite construction on an uncountable well ordered index set (of an inverse system) as well as on  $(J, \leq)$  (see [25]).

## Conflict of interest

The author declares no competing financial or any other selfish interest.

## References

- [1] Borsuk K. Concerning homotopy properties of compacta. *Fundamenta Mathematicae*. 1968; 62: 223-254.
- [2] Borsuk K. *Theory of shape, lecture notes series 28*. Matematisk Institute, Aarhus University; 1971.
- [3] Mardešić S, Segal J. Shapes of compacta and ANR-systems. *Fundamenta Mathematicae*. 1971; 72: 41-59.
- [4] Chapman TA. On some applications of infinite-dimensional manifolds in the theory of shape. *Fundamenta Mathematicae*. 1972; 76: 261-276.
- [5] Chapman TA. Shapes of finite-dimensional compacta. *Fundamenta Mathematicae*. 1972; 76: 181-193.
- [6] Mardešić S. Shapes for topological spaces. *General Topology and its Applications*. 1973; 3: 265-282.
- [7] Keesling J. *Products in the shape category and some applications*. Symposia Mathematica. Academic Press, New York; 1974. p.133-142.
- [8] Morita K. The Hurewicz and the Whitehead theorems in shape theory. *Science Reports of the Tokyo Kyoiku Daigaku, Section A*. 1974; 12: 246-258.
- [9] Morita K. On shapes of topological spaces. *Fundamenta Mathematicae*. 1975; 86: 251-259.
- [10] Mardešić S. Inverse limits and resolutions. *Shape Theory and Geometric Topology*. Lecture notes in mathematics. Springer-Verlag, Berlin; 1981. p.239-252.
- [11] Borsuk K. *Theory of Shape*. Monografie Matematyczne 59. Polish Scientific Publishers, Warszawa; 1975.
- [12] Dydak J, Segal J. *Shape theory: An introduction*. Lecture Notes in Math. Springer-Verlag, Berlin; 1978.
- [13] Mardešić S, Segal J. *Shape Theory*. North Holland, Amsterdam; 1982.
- [14] Cordier JM, Porter T. *Shape theory: Categorical methods of approximation*. Ellis Horwood Ltd., Chichester; 1989.
- [15] Quillen DG. *Homotopical Algebra*. Lecture notes in math. Springer-Verlag, Berlin; 1967.
- [16] Edwards DA, Hastings HM. *Čech and Steenrod homotopy theories with applications to geometric topology*. Lecture Notes in Math. Springer-Verlag, Berlin; 1976.
- [17] Cathey FW. *Strong Shape Theory*. Thesis, University of Washington; 1979.
- [18] Dydak J, Segal J. *Strong shape theory*. Dissertationes Mathematica; 1981.
- [19] Borsuk K. Some quantitative properties of shapes. *Fundamenta Mathematicae*. 1976; 93: 197-212.
- [20] Mardešić S. Comparing breast in a shape bration. *Glasnik Matematicki*. 1978; 13(33): 317-333.

- [21] Keesling J, Mardešić S. A shape fibration with bars of different shape. *Pacific Journal of Mathematics*. 1979; 84: 319-331.
- [22] Goodearl KR, Rushing TB. Direct limit groups and the Keesling-Mardešić shape fibration. *Pacific Journal of Mathematics*. 1980; 86: 471-476.
- [23] Mardešić S, Uglešić N, A category whose isomorphisms induce an equivalence relation coarser than shape. *Topology and its Applications*. 2005; 153: 448-463.
- [24] Uglešić N. Continuity in the coarse and weak shape categories. *Mediterranean Journal of Mathematics*. 2012; 9: 741-766.
- [25] Uglešić N. An example relating the coarse and weak shape. *Mediterranean Journal of Mathematics*. 2016; 13: 4939-4947.
- [26] Koceić BN, Uglešić N. The coarse shape. *Glasnik Matematički*. 2007; 42(62): 145-187.
- [27] Koceić BN. On some coarse shape invariants. *Topology and its Applications*. 2010; 157: 2679-2685.
- [28] Uglešić N. The shapes in a concrete category. *Glasnik Matematički*. 2016; 51(71): 255-306.
- [29] Herlich H, Strecker GE. *Category Theory*. Allyn and Bacon Inc., Boston; 1973.